LECTURE 26 Buckling of Columns. Inelastic Buckling (Part 2) Short version

1 Yasinsky's Formula

For a column stressed beyond the elastic limit the critical force is considered as linear function of the slenderness ratio:

$$F_{cr} = (a - b\lambda)A, \tag{1}$$

where *a* and *b* are coefficients, the values of which are given in the Table 1:

Table 1

Material	*a*, MPa	*b*, MPa
Low carbon steel	310	1.14
High carbon steel	469	2.62
Stainless steel	1000	5.4
Aluminum alloy	380	2.185
Cast iron	776	1.20
Wood (pine)	40	0.203

Formula (1) is called **Yasinsky's formula**.
According to both considered above formulae

$$\sigma_{cr} = \frac{\pi^2 E}{\lambda^2}, \quad \sigma_{cr} < \sigma_{pr} \quad (\lambda > \lambda_{\lim}) \tag{2}$$

and

$$\sigma_{cr} = a - b\lambda, \quad \sigma_{cr} > \sigma_{pr} \quad (\lambda < \lambda_{\lim}). \tag{3}$$

Let us plot the diagram

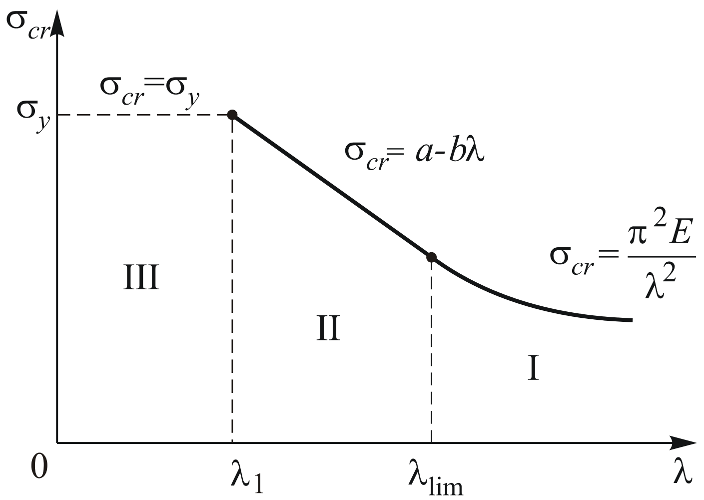

Fig. 1

and classify the columns as the following:

(I) $\lambda > \lambda_{\lim}$ (**long columns**)

$$\sigma_{cr} = \frac{\pi^2 E}{\lambda^2}, \tag{4}$$

$$F_{cr} = \frac{\pi^2 E I_{\min}}{(\nu l)^2}, \tag{5}$$

(II) $\lambda < \lambda_{\lim}$ (**intermediate columns**)

$$\sigma_{cr} = a - b\lambda, \tag{6}$$

$$F_{cr} = (a - b\lambda)A, \tag{7}$$

(III) $\lambda < \lambda_1$ (**short columns**)

$$\sigma_{cr} = \sigma_y, \tag{8}$$

$$F_{cr} = \sigma_y A. \tag{9}$$

Let us note, that the curve $\sigma_{cr} = f(\lambda)$ plotted above determines the value of limiting stress. The actual working stresses in a structure must be less than σ_{cr}.

For short column (small λ) the ratio of σ_{cr} to the largest working (allowable) stress represents merely the factor of safety

$$\frac{\sigma_{cr}}{[\sigma]_c} = n = \frac{\sigma_y}{[\sigma]_c}, \tag{10}$$

where $[\sigma]_c$ – is allowable stress in compression. At large values of λ this ratio is called the **stability factor of safety**

$$\frac{\sigma_{cr}}{[\sigma]_s} = n_s, \tag{11}$$

where $[\sigma]_s$ – allowable stress in buckling.

Denote

$$\frac{\sigma_{cr}}{\sigma_y} = \varphi, \tag{12}$$

then

$$\sigma_{cr} = \varphi\sigma_y \tag{13}$$

or

$$[\sigma]_s = \varphi[\sigma]_c, \tag{14}$$

where φ is the **allowable stress reduction factor**. It is evident, that $\varphi < 1$ to prevent possible buckling of compressed posts.

In the design of metal and timber structures it is now legitimate practice to analyze columns on the basis of the allowable stress reduction factor. The calculation is carried out as for a short member in compression but the allowable stress is taken reduced, i.e. equal to $\varphi[\sigma]_c$, instead of $[\sigma]_c$, where φ is the ***allowable stress reduction factor depending on the slenderness ratio*** λ. The values of the factor φ are given in the Table 2:

Table 2

	Carbon steel	High carbon steel	Cast iron	Wood
λ	φ	φ	φ	φ
0	1.00	1.00	1.00	1.00
10	0.99	0.98	0.97	0.99
20	0.96	0.95	0.91	0.97
30	0.94	0.92	0.81	0.93
40	0.92	0.89	0.69	0.87
50	0.89	0.86	0.57	0.80
60	0.86	0.82	0.44	0.71
70	0.81	0.76	0.34	0.60
80	0.75	0.70	0.26	0.48
90	0.69	0.62	0.20	0.38
100	0.60	0.51	0.16	0.31
110	0.52	0.43		0.25
120	0.45	0.36		0.22
130	0.40	0.33		0.18
140	0.36	0.29		0.16
150	0.32	0.26		0.14
160	0.29	0.24		0.12
170	0.26	0.21		0.11
180	0.23	0.19		0.10
190	0.21	0.17		0.09
200	0.19	0.15		0.08

2 Condition of Stability

$$\sigma = \frac{F}{A} \le \varphi[\sigma]_c . \qquad (15)$$

This condition makes it possible to solve three problems as follows:

(1) ***check the stability*** of a bar for the specified load and cross-sectional area:

$$\sigma = \frac{F}{A} \le \varphi[\sigma]_c \qquad (16)$$

(2) ***determine the allowable load*** on a bar by the specified cross-sectional area and allowable stress

$$[F] \le A\varphi[\sigma]_c \qquad (17)$$

(3) ***determine the cross-sectional area*** A for the specified load and allowable stress $[\sigma]$.

3 Examples

Example 1

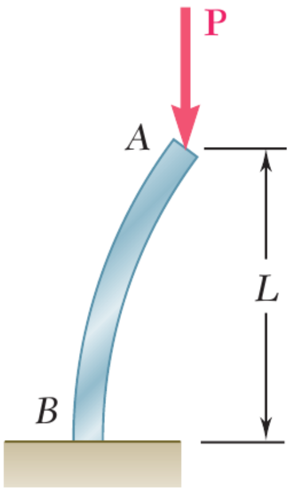

Fig. 2

Given: the length $l = 3$ m, the material is steel CT.3, The cross-section is I-section No16 (see assortments), $i_{\min} = 1.9$ cm, $I_{\min} = 77.6\,\text{cm}^4$, $A = 21.5\,\text{cm}^2$, the modulus of elasticity $E = 2\times10^{11}$ Pa, the proportional limit $\sigma_{pr} = 2\times10^2$ MPa, the coefficient of length reduction (**length reduction factor**) for a bar with built-in support $\nu = 2$.

It is necessary to determine the critical force F_{cr}.

Solution

(1) Calculate actual slenderness ratio: $\lambda = \dfrac{\nu l}{i_{\min}} = 316$.

(2) Determine the limiting slenderness ratio: $\lambda_{\lim} = \pi\sqrt{\dfrac{E}{\sigma_p}} = 100$.

(3) Because $\lambda > \lambda_{\lim}$, we will use Euler's formula to calculate the critical force:

$$F_{cr} = \frac{\pi^2 EI_{\min}}{(\nu l)^2} = 3.4\,\text{kN}.$$

Example 2

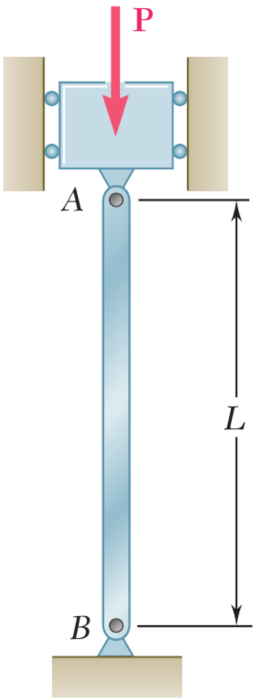

Fig. 3

Given: the length $l = 2\,\text{m}$, the material is wood, the rectangular cross section with the width $b = 5\,\text{cm}$, and height $h = 10\,\text{cm}$, the allowable compressive stress is $[\sigma]_c = 10\,\text{MPa}$, the force $P = 20\,\text{kN}$, $\nu = 1$.

Goal: Check the stability of the post.

Solution

$$i_{\min} = \sqrt{\frac{I_{\min}}{A}} = \sqrt{\frac{hb^3}{12bh}} = 1.43\ \text{cm},$$

$$\lambda = \frac{\nu l}{i_{\min}} = \frac{1 \times 200}{1.43} = 142.$$

The coefficient φ can be found in table by interpolation:

with $\lambda_1 = 150$ $\varphi = 0.14$

with $\lambda_2 = 140$ $\varphi = 0.16$

$$\varphi /_{\lambda = 142} = 0.16 - \frac{(0.16 - 0.14)}{10} \times 2 = 0.156,$$

Condition of stability is

$$\sigma = \frac{F}{A} \le \varphi[\sigma]_c,$$

$$\frac{20000}{5 \times 10^{-2} \times 10 \times 10^{-2}} \le 0.156 \times 10 \times 10^6,$$

$$4 \times 10^6 > 1.56 \times 10^6.$$

Thus the force $P = 20\,\text{kN}$ doesn't ensure the stability of the bar.

Example 3

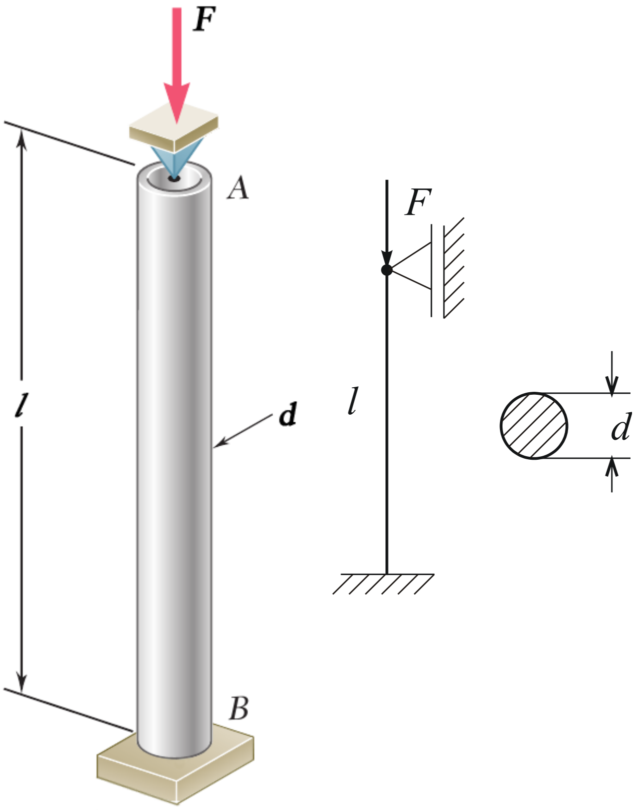

Fig. 4

Given: $[\sigma]_c = 160\,\text{MPa}$, $l = 2\,\text{m}$, $d = 10\,\text{cm}$, $\nu = 0.7$.

Goal: calculate allowable load $[F]$

Solution

(1) Determine actual slenderness ratio value: $\lambda = \dfrac{\nu l}{i_{\min}}$, where

$$i_{\min} = \sqrt{\frac{I_{\min}}{A}} = \sqrt{\frac{\pi d^4 \times 4}{64 \times \pi d^2}} = \frac{d}{4} = 2.5\,\text{cm}.$$

Substituting the numerical values, we get

$$\lambda = \frac{0.7 \times 200}{2.5} = 56.$$

(2) The coefficient $\varphi = \varphi(\lambda)$ (in table)

$$\varphi\big/_{\lambda=56} = 0.86 - \frac{0.86 - 0.82}{10} \times 6 = 0.836.$$

(3) Allowable force is

$$[F] = \varphi[\sigma]A = 0.836 \times 160 \times 10^6 \times \frac{3.14 \times (10 \times 10^{-2})^2}{4} = 1043.6\,\text{kN}.$$

Example 4

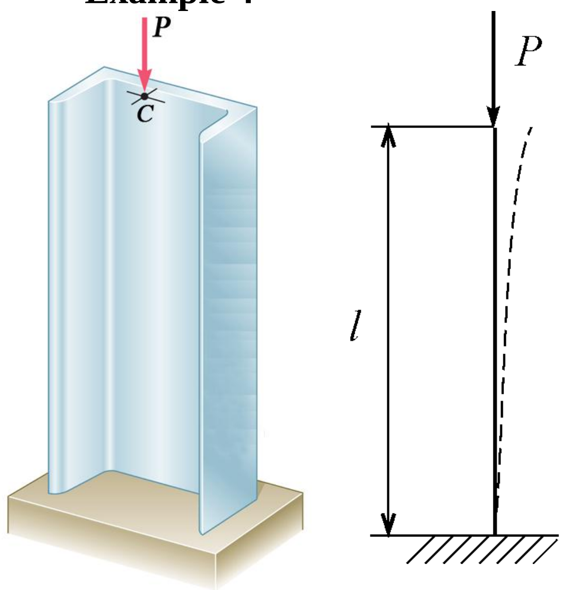

Fig. 5

Given:

$P = 200\,\text{kN}$, $l = 2\,\text{m}$, $[\sigma_c] = 160\,\text{MPa}$, $\nu = 2$, Steel 3.

Goal:

Number of channel section?

1-st iteration:

(1) Let us assume that $\varphi^I = 0.5$, since $0 < \varphi < 1$.

(2) $A^{I}=\frac{P}{\varphi^{I}[\sigma]_{c}}=\frac{2\times10^{5}}{0.5\times160\times10^{6}}=25\times10^{-4}\,\text{m}^2.$

(3) Nearest channel section from assortment is No20[a]: $i_{\min}=2.34\times10^{-2}$ m.

(4) For this section: $\lambda^{I}=\frac{\nu l}{i_{\min}}=2\frac{2}{2.34\cdot10^{-2}}=170.$

(5) Stress reduction factor for the rod with actual slenderness ratio 170 is $\varphi^{I*}=0,26$ (see Table 2).

2-nd iteration:

(1) Let us accept $\varphi^{II}=\frac{\left(\varphi^{I}+\varphi^{I*}\right)}{2}=0.38$.

(2) Substituting this value of stress reduction factor into condition of stability we get new value of cross-sectional area: $A^{II}=\frac{200\times10^{3}}{0.38\times160\times10^{6}}=33.4\times10^{-4}\,\text{m}^2.$

(3) From assortment, find corresponding number of channel: No27 with $A=35.2\times10^{-4}\,\text{m}^2$ and $i_{\min}^{II}=2.73\times10^{-2}$ m.

(4) For No27 channel, actual slenderness ratio is $\lambda^{II}=\frac{\nu l}{i_{\min}^{II}}=146$ and stress reduction factor $\varphi^{II*}=0.34$.

(5) Since difference between $\varphi^{II}=0.38$ and $\varphi^{II*}=0.34$ is more than 5% ($\Delta=\left(0.38-0.34\right)/0.38=0.105=10.5\%$) we must consider next iteration assuming $\varphi^{III}=\frac{0.34+0.38}{2}=0.36$.

(6) Substituting this stress reduction factor into condition of rigidity, we find $A^{III}=35\times10^{-4}\,\text{m}^2$. It corresponds to the channel No27. It means that final result of our calculation is channel No27.

MINISTRY OF EDUCATION AND SCIENCE OF UKRAINE

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Course
Mechanics of materials and structures

HOME PROBLEM 18

Buckling and Stability of Compressed Rods

Name of student:

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National aerospace university
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Department of aircraft strength

Subject: mechanics of materials
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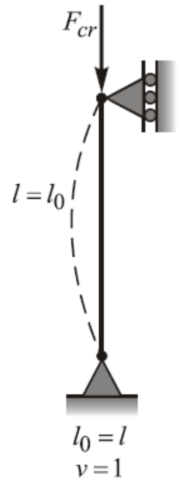
Topic: Buckling and Stability of Compressed Rods.

Full name of the student, group

Variant: 1 **Complexity: 1**



Given: $l = 3$ m, cross-section: I-beam, $F = 100$ kN.

Goal: 1) determine the cross-sectional dimensions; 2) calculate the value of critical force for selected column; 3) calculate the value of allowable load for selected column.

signature

Full name of the lecturer

Mark:

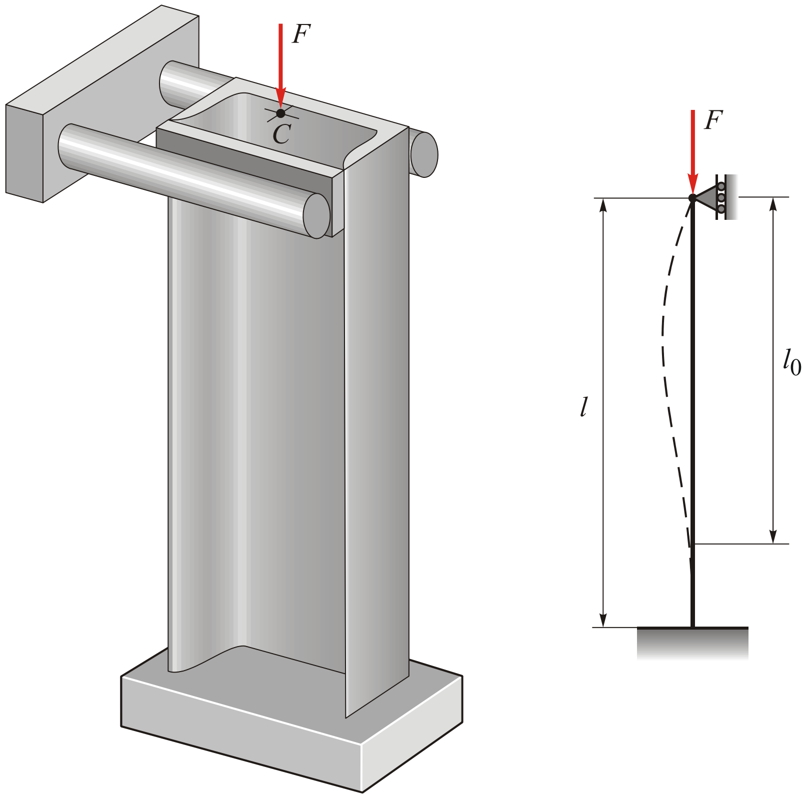


Fig. 1

Data: cross-section type – channel (see Fig. 1), $l = 2.5\,\text{m}$, $\sigma_{pr} = 250\,\text{MPa}$, $[\sigma]_c = 160\,\text{MPa}$, $F = 150\,\text{kN}$.

Goal:

1) Determine channel section number from the condition of stability.
2) For selected number, calculate critical force value F_{cr}.
3) Determine actual value of the safety factor n_y
4) For selected number, calculate allowable force value $[F]$.

Solution

1. Selecting the channel section number from the condition of stability.

Condition of stability is the following:

$$\sigma = \frac{F}{A} \le [\sigma]_s, \text{ where}$$

$[\sigma]_s$ – allowable stress for stability:

$$[\sigma]_s = \varphi [\sigma]_c, \text{ where}$$

φ – stress reduction factor, A – unknown cross-sectional area, F – compressive force.

Since φ factor is the tabular function of the post slenderness ratio λ, and the last value is determined by unknown cross-sectional dimensions, this problem will be solved by approximation method using available φ range: $0 < \varphi < 1$. In the first approach, we will assume that $\varphi^I = 0.5$.

I-st iteration

Calculating the area from the condition of stability:

$$A^I \ge \frac{F}{\varphi^I [\sigma]_c} = \frac{150 \times 10^3}{0.5 \times 160 \times 10^6} = 18.75 \times 10^{-4}\,\text{m}^2 = 18.75\,\text{cm}^2.$$

Applying the channel section assortment, let us determine the closest number of channel section and find it’s minimal geometrical properties, i.e. the moment and radius of inertia in the plain of maximum slenderness:

channel No 16: $A = 18.1\,\text{cm}^2$, $I_{\min} = 63.3\,\text{cm}^4$, $i_{\min} = 1.87\,\text{cm}$.

Knowing the length reduction factor of the post $\nu = 0.7$, let us calculate it's maximum slenderness ratio:

$$\lambda_{\max}^{I} = \frac{\nu l}{i_{\min}} = \frac{0.7 \times 2.5}{1.87 \times 10^{-2}} = 93.6.$$

Using this value in the table of stress reduction factor dependence on slenderness ratio, we will determine actual stress reduction factor value applying linear interpolation:

$$\varphi|_{\lambda=90} = 0.69, \quad \varphi|_{\lambda=100} = 0.60, \quad \varphi|_{\lambda=93.6} = 0.69 - \frac{0.69 - 0.60}{10} \times 3.6 = 0.6576.$$

After calculation, new value of the stress reduction factor becomes equal

$$\varphi^{I*} = 0.6576.$$

Comparing $\varphi^{I} = 0.5$ and $\varphi^{I*} = 0.6576$, we conclude that the error is:

$$\Delta = \frac{\varphi^{I*} - \varphi^{I}}{\varphi^{I}} \times 100\% = 31.52\% > 5\%.$$

Since the error exceeds 5% range ($\Delta > 5\%$), it will be necessary to perform second approach assuming

$$\varphi^{II} = \frac{\varphi^{I} + \varphi^{I*}}{2} = \frac{0.5 + 0.6576}{2} = 0.5788.$$

II-nd iteration

Cross-sectional area in this approach is

$$A^{II} = \frac{F}{\varphi^{II}[\sigma]_c} = \frac{150 \times 10^{3}}{0.5788 \times 160 \times 10^{6}} = 16.2\,\text{cm}^2.$$

The closest channel number is No 14: $A^{II} = 15.6\,\text{cm}^2$, $I_{\min} = 45.4\,\text{cm}^4$, $i_{\min} = 1.7\,\text{cm}$.

It's maximum slenderness ratio is:

$$\lambda_{\max}^{II} = \frac{\nu l}{i_{\min}} = \frac{0.7 \times 2.5}{1.7 \times 10^{-2}} = 102.9.$$

Applying the table of stress reduction factor dependence on slenderness ratio, we will determine actual stress reduction factor value also applying linear interpolation:

$$\varphi|_{\lambda=100} = 0.60, \quad \varphi|_{\lambda=110} = 0.52,$$

$$\varphi|_{\lambda=102.9} = 0.60 - \frac{0.60 - 0.52}{10} \times 2.9 = 0.5768 = \varphi^{II*}.$$

Comparing new (φ^{II*}) and old (φ^{II}) values shows that

$$\Delta = \frac{\varphi^{II*} - \varphi^{II}}{\varphi^{II}} \times 100\% = \frac{0.5768 - 0.5788}{0.5788} \times 100\% = 0.35\% < 5\%.$$

Due to $\Delta < 5\%$, our solution is successful. Selected channel No 14 has the following geometrical properties:

$$A = 15.6\,\text{cm}^2,\ I_{\min} = 45.4\,\text{cm}^4,\ i_{\min} = 1.7\,\text{cm},\ \lambda_{\max} = 102.9,\ \varphi = 0.5768.$$

2. Calculating the critical force value of the No 14 channel post.

First of all, let us determine limiting slenderness ratio of the post material (carbon steel):

$$\lambda_{\lim} = \pi\sqrt{\frac{E}{\sigma_{pr}}} = 3.14\sqrt{\frac{2\times10^{11}}{250\times10^{6}}} = 88.8.$$

Since $\lambda_{\max} > \lambda_{\lim}$ $(102.9 > 88.8)$, Euler's formula will be used for critical force calculating:

$$F_{cr} = \frac{\pi^2 E I_{\min}}{(\nu l)^2} = \frac{3.14^2\times2\times10^{11}\times45.4\times10^{-8}}{(0.7\times2.5)^2} = 292.6\,\text{kN}.$$

3. Calculating actual factor of safety for selected post No 14.

$$n_{st} = \frac{F_{cr}}{F} = \frac{292.6}{150} = 1.95.$$

4. Calculating the allowable value of external force for No 14 channel post.
For this purpose, we will use the condition of stability

$$\sigma = \frac{F}{A} \le \varphi[\sigma]_c.$$

Therefore, $[F] = A\varphi[\sigma]_c = 15.6\times10^{-4}\times0.5768\times160\times10^{6} = 144\,\text{kN}.$

Conclusion

Calculated actual factor of safety for selected post $n_{st} = 1.95$ exceeds the value of safety factor in compression ($n = 1.5 - 1.7$) due to high danger of buckling failure. This fact is the result of corresponding φ factor selection in regulation documents of civil engineering.