LECTURE 14 Strength of a Bar in Transverse Bending

1 Introduction

As we have seen, only normal stresses occur at cross sections of a rod in pure bending. The corresponding internal forces give the resultant bending moment at the section.

In transverse bending, a shearing force Q_z occurs at a section of a rod as well as a bending moment. This force represents the resultant of elementary distributed forces lying in the plane of the section. Consequently, in this case ***not only normal, but also shearing stresses occur at cross section of a rod***:

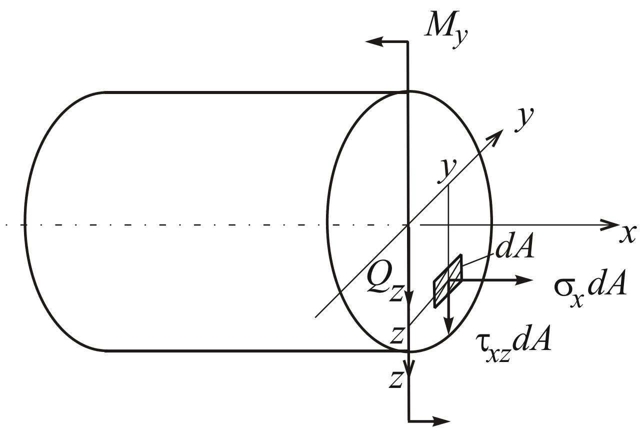


Fig. 1

$$M_y = \int_A \sigma_x z dA, \qquad (1)$$

$$Q_z = \int_A \tau_{xz} dA. \qquad (2)$$

The appearance of ***shearing stresses*** τ is accompanied by ***shearing strains*** γ. Hence any elementary section dA undergoes angular displacement due to shear. The ***magnitude of normal stresses is not, however, sensibly affected by the shearing forces***. When the shearing force varies along the rod axis the formula of pure bending introduces an error in the calculation of σ_x:

$$\sigma_x(z) = \frac{M_y z}{I_y} \qquad (3)$$

By a simple analysis it may be shown that this error is of the order of h/l compared to unity, where h is the dimension of the cross section in the plane of bending and l is the length of a rod.

By definition the characteristic feature of a rod is that the dimensions its cross section are much smaller than the length. Consequently, the quantity h/l is very small, and so is the error involved.

2 Determination of Shearing Stresses

Let us estimate the shearing stresses in transverse bending. In general case of external loading we have the element of the rod *dx* in equilibrium:

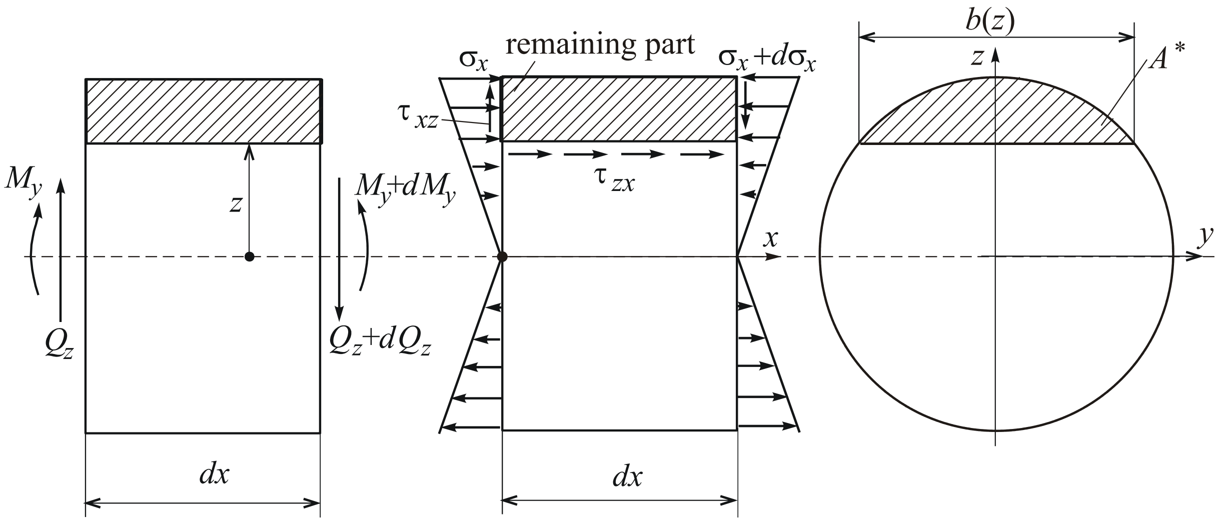


Fig. 2

Isolate an element of length *dx* from the rod. Divide the element in two parts by longitudinal horizontal plane (layer) passed at a distance *z* from neutral layer and consider the conditions of equilibrium of the upper part:

$$\sum F_x = 0: \quad \int_{A^*} \sigma_x dA + \tau_{zx} b(z)dx - \int_{A^*} \left(\sigma_x + d\sigma_x\right) dA = 0. \tag{4}$$

We assume that ***the shearing stresses are uniformly distributed across the width of the layer $b(z)$***:

$$\int_{A^*} \sigma_x dA + \tau_{zx} \cdot b(z)dx - \int_{A^*} \sigma_x dA - \int_{A^*} d\sigma_x dA = 0,$$

$$\tau_{zx} b(z)dx = \int_{A^*} d\sigma_x dA, \quad d\sigma_x = \frac{dM_y}{I_y} z,$$

$$\tau_{zx} b(z)dx = \frac{dM_y}{I_y} \int_{A^*} z dA,$$

where

$$\int_{A^*} z dA = S_{n.a.}^* = S_y^*$$

is ***static(al) moment (first moment) with respect to the y central axis of the portion of the area located above (or below) the z layer***:

$$\tau_{zx} b(z)dx = \frac{dM_y S_y^*}{I_y}.$$

Since

$$\frac{dM_y}{dx} = Q_z,$$

we get

$$\tau_{xz}(z) = \tau_{zx}(z) = \frac{Q_z S_y^*}{I_y b(z)} \quad \text{– \textbf{Juravsky formula}.} \tag{5}$$

3 Examples of Shear Stress Distribution in Different Cross Sections

Example 1 Shear stress distribution along a height of rectangle

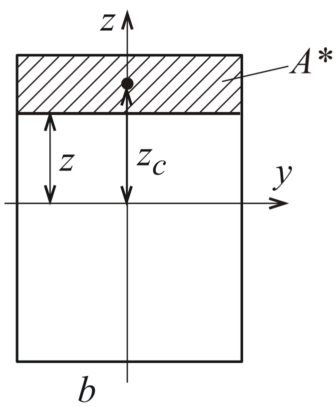


Fig. 3

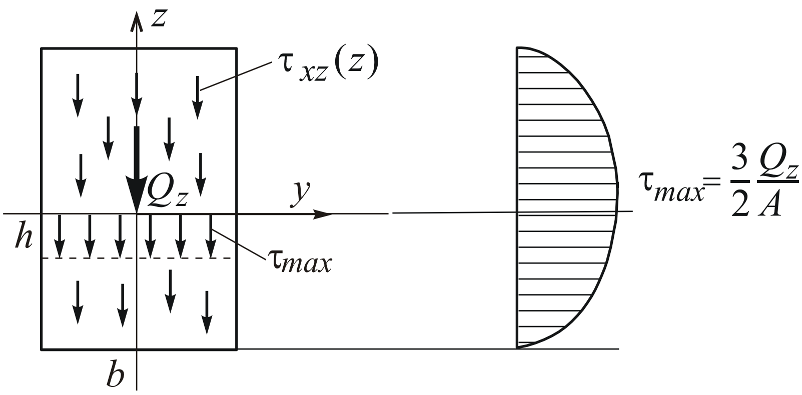


Fig. 4

$$b(z)=b,\quad I_y=\frac{bh^3}{12},\quad S_y{*}=A{*}z_c=\left(\frac{h}{2}-z\right)b\left(\frac{h}{2}+z\right)\frac{1}{2}=\frac{1}{2}\left(\frac{h^2}{4}-z^2\right)b,$$

$$\tau(z)=\frac{Q_z b\frac{1}{2}\left(\frac{h^2}{4}-z^2\right)12}{bh^3 b}=\frac{6Q_z}{bh^3}\left(\frac{h^2}{4}-z^2\right)\Bigg|_{z=0}=\frac{3}{2}\frac{Q_z}{A}\Bigg|_{z=\frac{h}{2}}=0.$$

Note, that maximum stress occurs at $z=0$ and are distributed uniformly along y axis.

Example 2 Shear stress distribution along a height of round cross section

In this case we have:

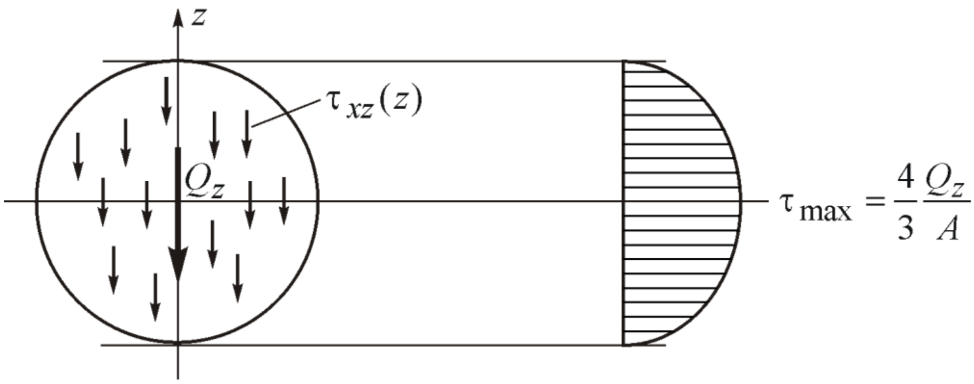


Fig.5

4 Comparison of Normal and Shearing Stresses in Bending

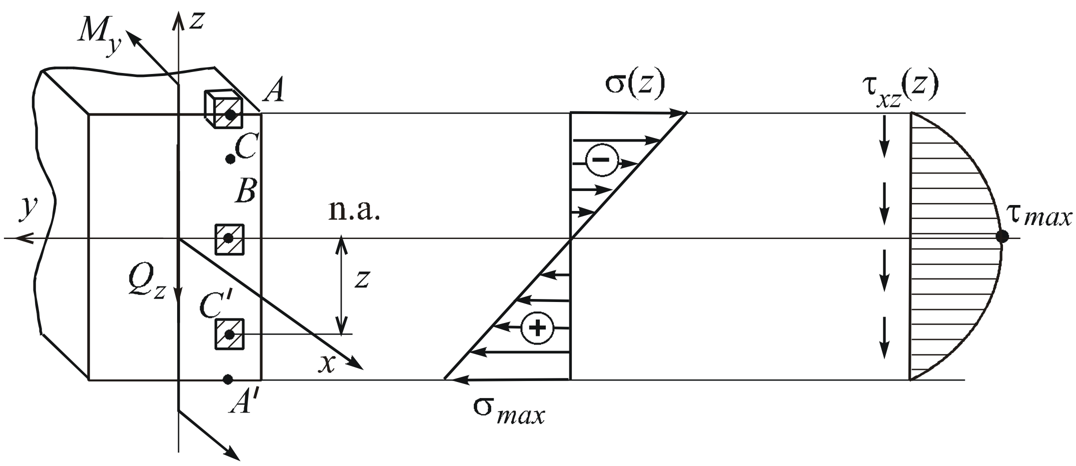


Fig. 6

1. Points A and A'.

These points are most remote from the neutral axis. ***Shearing stresses are zero***:

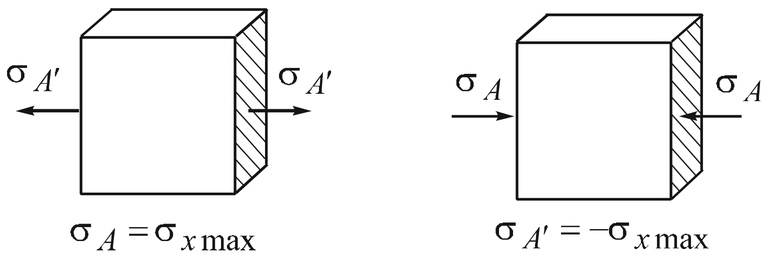


Fig.7

2. Point *B*. $\sigma_x = 0$. ***Shearing stresses are maximum***:

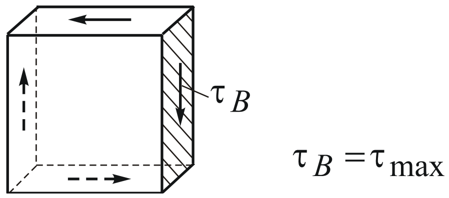


Fig.8

3. Points *C* and *C′*.

There is biaxial stress state at these points:

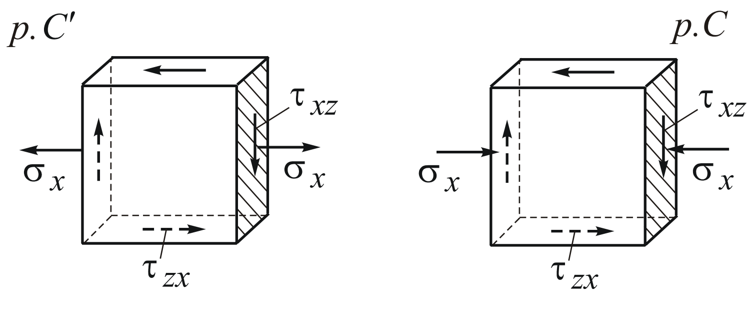


Fig. 9

Let us determine the principal stresses using general formula:

$$\sigma_{1,\,2\,(3)} = \frac{\sigma_x + \sigma_z}{2} \pm \frac{1}{2}\sqrt{\left(\sigma_x - \sigma_z\right)^2 + 4\tau_{xz}^{\;2}}\,.$$

Since $\sigma_z = 0$,

$$\sigma_1 = \frac{\sigma_x}{2} + \frac{1}{2}\sqrt{\sigma_x^{\;2} + 4\tau_{xz}^{\;2}}\,,$$

$$\sigma_2 = 0\,,$$

$$\sigma_3 = \frac{\sigma_x}{2} - \frac{1}{2}\sqrt{\sigma_x^{\;2} + 4\tau_{xz}^{\;2}}\,.$$

A comparison may be made of the absolute values of the maximum normal and maximum shearing stresses on cross section of a prismatic rod. For example, for a cantilever beam of rectangular section we have:

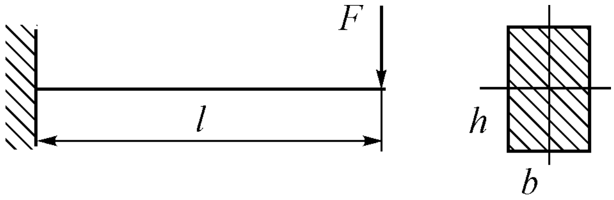


Fig. 10

$$\sigma_{\max}=\frac{M_{\max}}{W_{n.a.}}=\frac{6Fl}{bh^2}, \quad \tau_{\max}=\frac{3}{2}\frac{F}{bh},$$

whence

$$\frac{\tau_{\max}}{\sigma_{\max}}=\frac{h}{4l}.$$

This means that the maximum shearing stresses and the maximum normal stresses in the cross section are in about the same ratio as the cross-sectional dimension of the section and the length of the rod, i.e. the ***shearing stresses are appreciable less than the normal stresses for rectangle cross section***.

Because of the small magnitude of $\tau_{\max}$ the ***analysis of prismatic solid rods subjected to transverse bending is made only on the basis of normal stresses as in pure bending. Shearing stresses are not taken into consideration***.

Therefore, the condition of strength in transverse bending of solid prismatic bars is

$$\sigma_{\max}=\frac{M_{y\max}}{W_{n.a.(y)}}\le[\sigma]. \qquad (6)$$

5 Curvature of a Beam

When loads are applied to a beam, its longitudinal axis is deformed into a curve. As an example, consider a cantilever beam *AB* subjected to a load a *F* at the free end (Fig. 11):

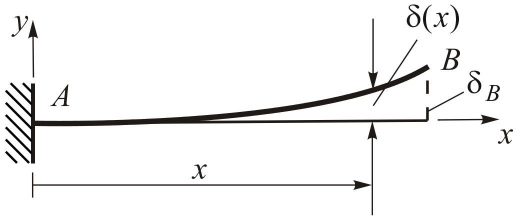


Fig. 11

The initially straight axis is bent into a curve, called the **deflection curve** of the beam.

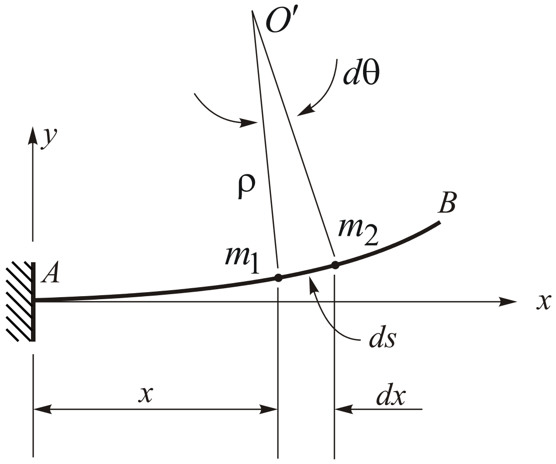
Let us construct a system of coordinate axes (x, y) with the origin located at a suitable point on the longitudinal axis of the beam. We will place the origin at the fixed support. The positive x axis is directed to the right, and the positive y axis is directed upward.

The beam considered is assumed to be symmetric about the xy plane, which means that the y axis is an axis of symmetry of the cross section. In addition, all loads must act in the xy plane. As a consequence, the bending deflections occur in this same plane, known as the plane of bending.

The **deflection** of the beam at any point along its axis is the displacement of that point from its original position, measured in the y direction. We denote the deflection by the letter δ. It is important to note that the resulting strains and stresses in the beam are directly related to the **curvature** of the deflection curve.

To illustrate the concept of curvature, consider again a cantilever beam subjected to a load F acting at the free end (Fig. 12). The deflection curve of this beam is shown at the bottom of the Fig. 12. For purposes of analysis, we will identify two points m_1 and m_2 on the deflection curve. Point m_1 is selected at an arbitrary distance x from the y axis and point m_2 is located at a small distance ds further along the curve. At each of this points we draw a line normal to the tangent to the deflection curve, that is, normal to the curve itself. This normals intersect at point O', which is the **center of curvature** of the deflection curve. Because most beams have very small deflections and nearly flat deflection curve, point O' is usually located much further from the beam that is indicated in the Fig. 12.

(a)



(b)

Fig. 12

The distance m_1O' from the curve to the centre of curvature is called the **radius of curvature** ρ, and the **curvature** k is defined as the reciprocal of the radius of curvature. Thus,

$$k = \frac{1}{\rho}. \tag{7}$$

Curvature is a measure of how sharply a beam is bent.

From the geometry of triangle $O'm_1m_2$ we obtain

$$\rho d\theta = ds \tag{8}$$

in which $d\theta$ (measured in radians) is the infinitesimal angle between the normals and ds is the infinitesimal distance along the curve between points m_1 and m_2. Combining Eqs (7) and (8), we get

$$k = \frac{1}{\rho} = \frac{d\theta}{ds}. \tag{9}$$

It is interesting to note, that if the curvature is constant throughout the length of a curve, the radius of curvature will also be constant and the curve will be an arc of a circle.

The deflections of a beam are usually very small compared to its length. Small deflections mean that the deflection curve is nearly flat. Consequently, the distance ds along the curve may be set equal to its horizontal projection dx (Fig. 12). Under this conditions the equation for the curvature becomes

$$k = \frac{1}{\rho} = \frac{d\theta}{dx}. \tag{10}$$

Both the curvature and the radius of curvature are functions of the distance x. It follows that the position O' of the center of curvature also depends upon the distance x.

We will see that the curvature at a particular point on the axis of a beam depends on the bending moment at that point and on the properties of the beam itself. Therefore, if the beam is prismatic and the material is homogeneous, the curvature will vary only with the bending moment. Consequently, a ***beam in pure bending will have constant curvature and the beam in transverse bending will have varying curvature***.

The **sign convention for curvature** depends on the orientation of the coordinate axes. If the x axis is positive to the right and the y axis is positive upward, then the ***curvature is positive when the beam is bent concave upward (or convex downward) and the center of curvature is above the beam***. Conversely, the curvature is negative when the beam is bent concave downward (or convex upward) and the center of curvature is below the beam. This sign convention is represented on the Fig. 13.

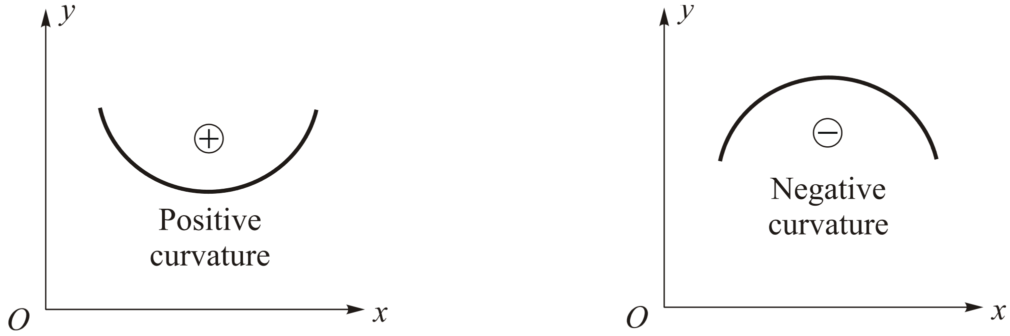


Fig. 13

6 Examples of practical problems solution

Example 1 Checking problem of cantilever beam

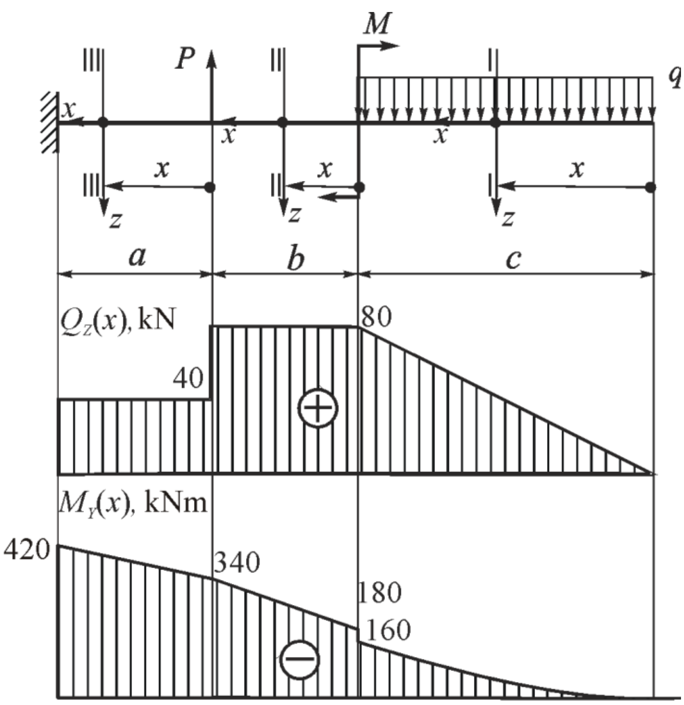


Fig.14

Given: $[\sigma]=160\,\text{MPa}$, rectangle cross-section: width $l=10\times10^{-2}\,\text{m}$, height $h=40\times10^{-2}\,\text{m}$, $a=2\,\text{m}$, $b=2\,\text{m}$, $c=4\,\text{m}$, $P=40\,\text{kN}$, $M=20\,\text{kNm}$, $q=20\,\text{kN/m}$.

R.D.: Check the strength of cantilever under specified loading.

Solution

1) Calculating the internal forces $Q_z(x)$ and $M_y(x)$, applying the method of sections.

I–I $0 \le x \le 4\,\text{m}$

$$Q_z{}^I(x) = qx\big|_{x=0} = 0\big|_{x=c} = 80\,\text{kN};$$

$$M_y{}^I(x) = \frac{-qx^2}{2}\Big|_{x=0} = 0\big|_{x=c} = -160\,\text{kNm}$$

II–II $0 \le x \le 2\,\text{m}$

$$Q_z{}^{II}(x) = qc = 80\,\text{kN};$$

$$M_y{}^{II}(x) = -qc\left(\frac{c}{2} + x\right) - M\Big|_{x=0} = -180\big|_{x=b} = -340\,\text{kNm};$$

III–III $0 \le x \le 2\,\text{m}$

$$Q_z{}^{III}(x) = qc - P = 40\,\text{kN};$$

$$M_y{}^{III}(x) = -qc\left(\frac{c}{2} + b + x\right) - M - Px\Big|_{x=0} = -340\big|_{x=a} = -420\,\text{kNm}.$$

2) To check the strength, we will write condition of strength for critical layer of critical cross-section, i.e. the cross-section with $\left|M_y\right|_{\max} = 420\,\text{kNm}$:

$$\sigma_{\max} = \frac{M_{y\max}}{W_{н.о}} \le [\sigma].$$

In our case,

$$\sigma_{\max} = \frac{6M_{y\max}}{lh^2} = \frac{6 \times 420 \times 10^3}{10 \times 10^{-2} \times 40^2 \times 10^{-4}} = 157.5\,\text{MPa}.$$

Since 157.5 < 160, the cantilever is strong.

3) Designing the graphs $\sigma(z)$ and $\tau(z)$ in critical cross-section with $\left|M_y\right|_{\max} = 420\,\text{kNm}$ and $\left|Q_z\right| = 40\,\text{kN}$ (see Fig. 15).

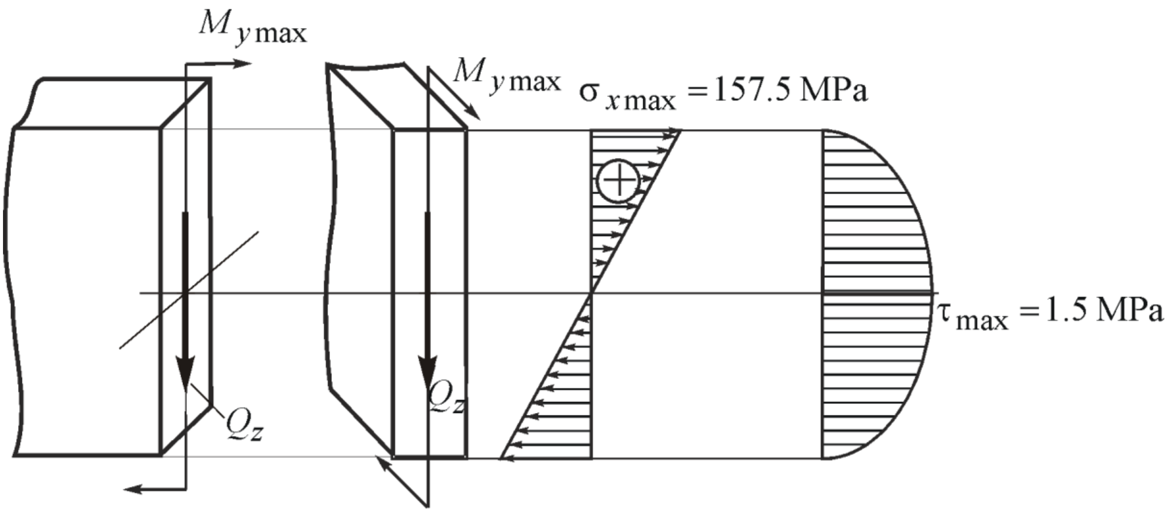


Fig. 15

$$\sigma(z) = \frac{M_{y\max} z}{I_y}; \quad \tau(z) = \frac{Q_z S_y(z)}{b I_y};$$

$$\tau_{\max} = \frac{Q_z b \frac{h}{2}\frac{h}{4}}{b\frac{bh^3}{12}} = \frac{3}{2}\frac{Q_z}{A} = \frac{3 \times 40 \times 10^3}{2 \times 1 \times 10^{-1} \times 4 \times 10^{-1}} = 1.5\,\text{MPa}.$$

Since $\tau_{\max} << \sigma_{\max}$ we will ignore shear stresses in stress analysis of prismatic beams.

4) Calculating maximum shear stresses in the beam.

In prismatic beams of constant cross-section maximum shear stress will act in the cross-section with $Q_{z\max} = 80\,\text{kN}$. Corresponding $M_y = 340\,\text{kNm}$. These internal forces are applied in Fig. 16.

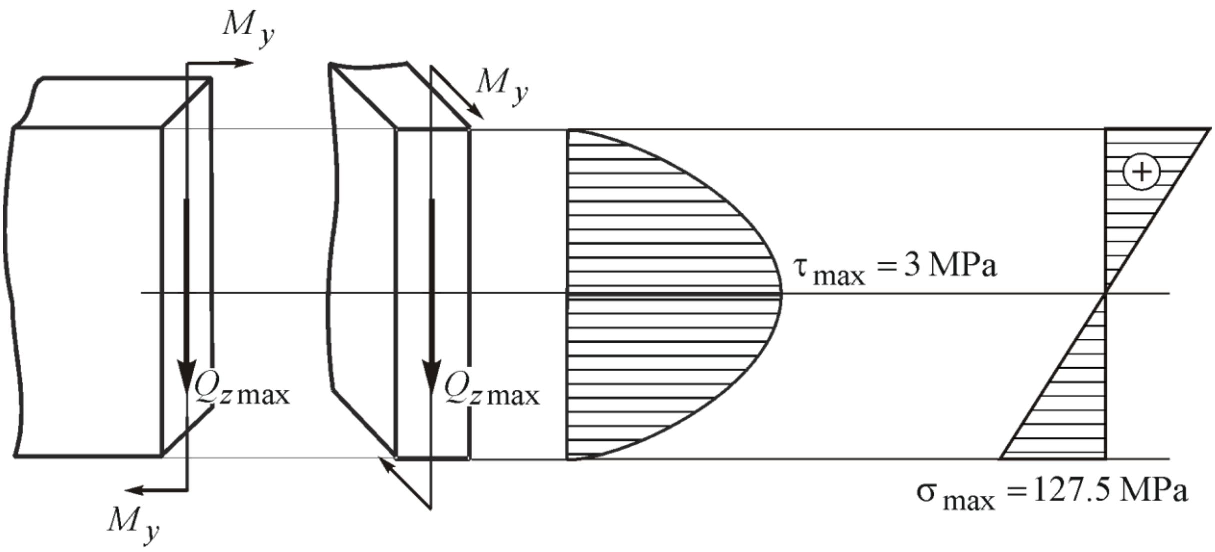


Fig.16

$$\tau_{\max} == \frac{3}{2}\frac{Q_{z\max}}{A} = \frac{3\times 80\times 10^3}{2\times 1\times 10^{-1}\times 4\times 10^{-1}} = 3\,\text{MPa}.$$

In this cross-section, normal stresses are calculated by the formula

$$\sigma_{\max} = \frac{6M_y}{lh^2} = \frac{6\times 340\times 10^3}{10\times 10^{-2}\times 40^2\times 10^{-4}} = 127.5\,\text{MPa}.$$

Conclusion. In stress analysis of prismatic beams, it's possible to ignore influence of shear stresses on the results of calculation.

Example 2 Design problem for cantilever beam

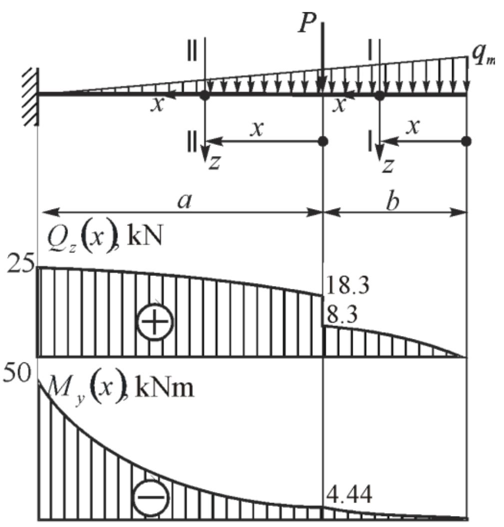


Fig. 17

Given: Steel cantilever is loaded by linearly distributed loading with maximum intensity $q_m = 10\,\text{kN/m}$ and concentrated force $P = 10\,\text{kN}$. Also, $[\sigma] = 160\,\text{MPa}$, $a = 2\,\text{m}$, $b = 1\text{m}$.

R.D.: 1) number of I-beam section;

2) dimensions of rectangle cross-section in $\frac{h}{b} = 2$;

3) diameter of round solid cross-section.

Solution

1) Designing the graphs of shear forces $Q_z(x)$ and bending moments $M_y(x)$, applying method of sections.

I–I $0 \le x \le 1\text{m}$

$$Q_z{}^{\text{I}}(x) = qx - \frac{qx^2}{2(a+b)}\Big|_{x=0} = 0\Big|_{x=b} = 8.3\,\text{kN};$$

$$M_y{}^{\text{I}}(x) = -\frac{qx^2}{2} + \frac{qx^3}{6(a+b)}\Big|_{x=0} = 0\Big|_{x=b} = -4.44\,\text{kNm}.$$

II–II $0 \le x \le 2\,\text{m}$

$$Q_z^{\text{II}}(x) = q(x+b) - q\frac{(x+b)^2}{2(a+b)} + P\Big|_{x=0} = 18.3\Big|_{x=a} = 25\,\text{kN};$$

$$M_y^{II}(x) = -\frac{q(x+b)^2}{2} + \frac{q(x+b)^3}{6(a+b)} - Px\Big|_{x=0} = -4.44\Big|_{x=a} = -50\,\text{kNm}.$$

It is clear that critical cross-section is in rigid support since $\left|M_y\right|_{\max} = 50\,\text{kNm}$.

2) Calculating the sectional modulus of cross-section from condition of strength:

$$\sigma_{\max} = \frac{\left|M_y\right|_{\max}}{W_y} \le [\sigma]; \quad W_y \ge \frac{\left|M_y\right|_{\max}}{[\sigma]} = \frac{50\times10^3}{160\times10^6} = 312.5\times10^{-6}\,\text{m}^3.$$

3) Determining the dimensions of I-beam section knowing that $W_y \ge 312.5\times10^{-6}\,\text{m}^3$.

From assortment of steel products let us find near situated I-beam №24 with $W_y = 289\times10^{-6}\,\text{m}^3$ and estimate its overstress:

$$\sigma_{\max} = \frac{\left|M_y\right|_{\max}}{W_y} = \frac{50\times10^3}{289\times10^{-6}} = 173\,\text{MPa}.$$

Overstress $\Delta = \frac{173-160}{160}\cdot100\% = 8.1\% > 5\%$. It is more than allowable overstress equals to 5%. This result requires to choose larger neighbor I-beam section $№24^A$ with sectional modulus $W_y = 317\times10^{-6}\,\text{m}^3$. It will be understressed since

$$\sigma_{\max} = \frac{\left|M_y\right|_{\max}}{W_y} = \frac{50\times10^3}{317\times10^{-6}} = 157.7\,\text{MPa}.$$

Percentage of understress $\Delta = \frac{160-157.7}{160}\times100\% = 1.4\%$,

The dimensions of I-beam section $№24^A$ are presented in the Table:

h	b	d	t	A cm^2	I_y cm^4	W_y cm^3	S_y cm
мм							
240	125	5.6	9.8	37.5	3800	317	178

4) Designing the graphs of stress distribution $\sigma_x(z)$ and $\tau(z)$ in critical cross-section of I-beam section №24^A.

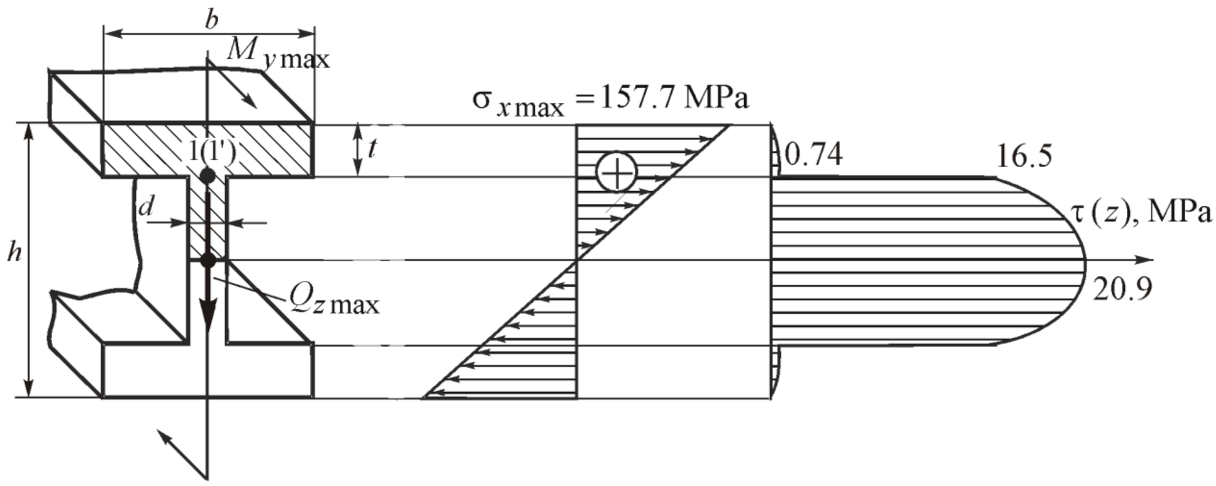


Fig. 18

To draw the graph of shear stress distribution it is necessary to use shear formula (Juravsky formula):

$$\tau(z) = \frac{Q_{z\max} S_y}{bI_y};$$

$$\tau_{\max} = \frac{Q_{z\max} S_y^{\text{т}}}{dI_y} = \frac{25\times10^3 \times 178\times10^{-6}}{5.6\times10^{-3}\times3800\times10^{-8}} = 20.9\,\text{MPa}.$$

Shear stress at the point 1 belonging to the I-beam flange is

$$\tau_{\text{p.1}} = \frac{Q_{z\max} bt\left(\frac{h}{2}-\frac{t}{2}\right)}{bI_y} = \frac{25\times10^3\times9.8\times10^{-3}\left(\frac{240}{2}-\frac{9.8}{2}\right)\times10^{-3}}{3800\times10^{-8}} = 0.74\,\text{MPa}.$$

Shear stress at the point 1′ belonging to the I-beam web is

$$\tau_{\text{p.1'}} = \frac{Q_{z\max} bt\left(\frac{h}{2}-\frac{t}{2}\right)}{dI_y} = \frac{25\times10^3\times125\times10^{-3}\times9.8\cdot10^{-3}\times\left(\frac{240}{2}-\frac{9.8}{2}\right)\times10^{-3}}{5.6\times10^{-3}\times3800\times10^{-8}} = 16.6\,\text{MPa}.$$

Corresponding graphs are shown in Fig. 18.

4) Determining the dimensions of rectangle section knowing that $W_y \geq 312.5\times10^{-6}\,\text{m}^3$.

$$W_y = \frac{bh^2}{6} \geq \frac{M_{y\max}}{[\sigma]} = \frac{50\times10^3}{160\times10^6} = 321.5\times10^{-6}\,\text{m}^3.$$

Substituting $\frac{h}{b} = 2$ we get

$$\frac{4}{6}b^3 \geq 321.5\cdot10^{-6} \quad \text{and} \quad b \geq \sqrt[3]{\frac{3\times321.5\times10^{-6}}{2}} = 7.8\times10^{-2}\,\text{m}$$

$$\text{and} \quad h = 2b = 15.6\times10^{-2}\,\text{m}.$$

Cross-sectional area of rectangle section is

$$A = bh = 7.8\times10^{-2}\times15.6\times10^{-2} = 121.8\times10^{-4}\,\text{m}^2.$$

The graphs of $\sigma(z)$ and $\tau(z)$ distribution in critical cross-section are shown in Fig. 19.

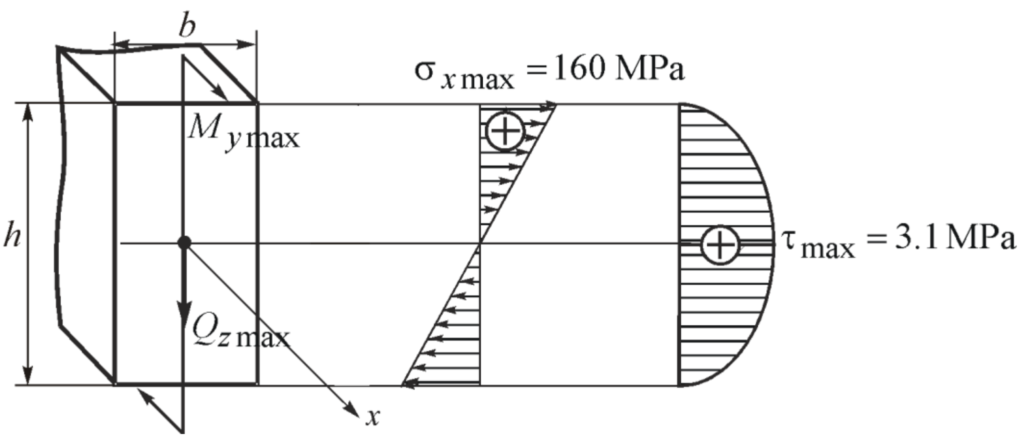


Fig. 19

$$\sigma_{\max} = \frac{M_{y\max}}{W_y} = \frac{6M_{y\max}}{bh^2} = \frac{6\times50\times10^3}{7.8\times10^{-2}\times(15.6)^2\times10^{-4}} = 158 \approx 160\,\text{MPa};$$

$$\tau_{\max} = \frac{3Q_{z\max}}{2A} = \frac{3\times25\times10^3}{2\times15.6\times7.8\times10^{-4}} = 3.1\text{MPa}.$$

5) Determining the dimensions of round section knowing that $W_y \geq 312.5\cdot10^{-6}\,\text{m}^3$.

$$W_y = \frac{\pi d^3}{32} \geq 312.5 \times 10^{-6}\,\text{m}^3.$$

$$d \geq \sqrt[3]{\frac{32W_y}{\pi}} = \sqrt[3]{\frac{32 \times 312.5 \times 10^{-6}}{3.14}} = 14.7 \times 10^{-2}\,\text{m}.$$

Cross-sectional area of round section is

$$\overset{\oslash}{A} = \frac{\pi d^2}{4} = \frac{3.14 \times \left(14.7 \times 10^{-2}\right)^2}{4} = 169.7 \cdot 10^{-4}\,\text{m}^2.$$

The graphs of $\sigma(z)$ and $\tau(z)$ distribution in critical cross-section are shown in Fig. 20.

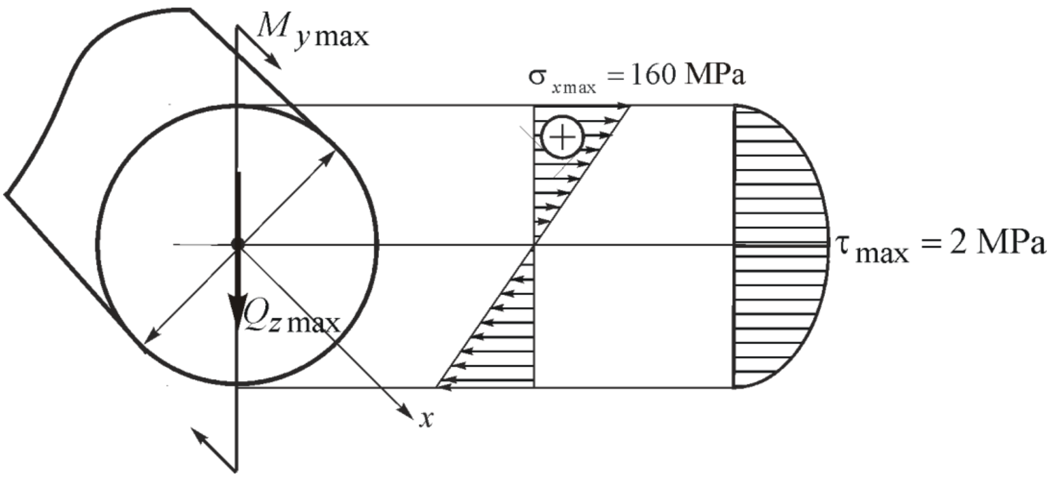


Fig. 20

$$\sigma_{\max} = \frac{\left|M_y\right|_{\max}}{W_y} = \frac{32 \times 50 \times 10^3}{3.14 \times \left(14.7 \times 10^{-2}\right)^3} = 160\,\text{MPa};$$

$$\tau_{\max} = \frac{Q_{z\max} S_y}{b I_y} = \frac{4}{3}\frac{Q_{z\max}}{A} = \frac{4 \times 25 \times 10^3}{3 \times 169.7 \times 10^{-4}} = 2\,\text{MPa}.$$

6) Comparing the cross-sectional areas:

$$\overset{I}{A} < \overset{\blacksquare}{A} < \overset{\oslash}{A} \rightarrow 37.5 \times 10^{-2} < 121.8 \times 10^{-2} < 169.7 \times 10^{-2}.$$

Note, that the I-beam section is the most effective in strength-to-weight ratio.

Example 3 Problem of allowable external load for two-supported beam

Given: steel simple beam with hollow rectangle cross-section is loaded by distributed load q and concentrated moment M in the left support. $H=10\,\text{cm}$, $B=6\,\text{cm}$, $h=4\,\text{cm}$, $b=2\,\text{cm}$, $[\sigma]=200\,\text{MPa}$, $a=1\,\text{m}$.

R.D.: the largest working (allowable) load $[q]$.

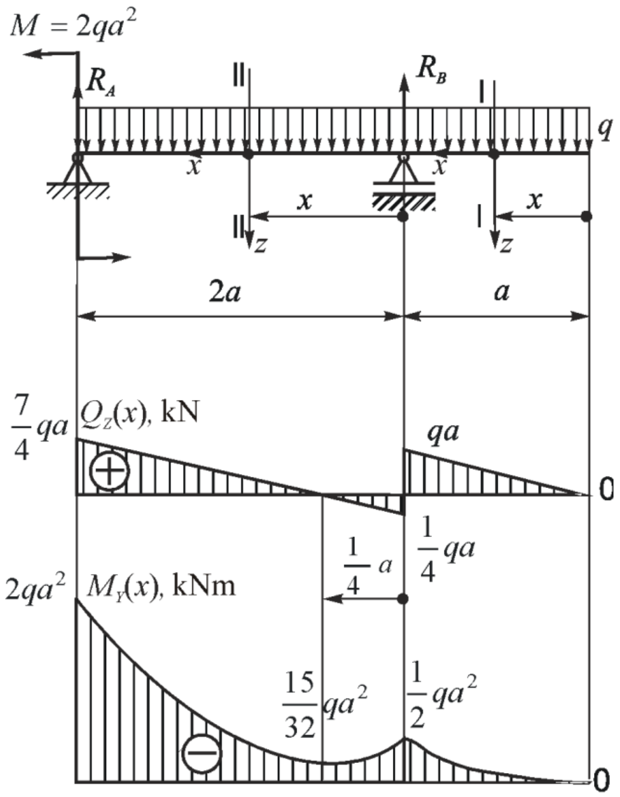


Fig. 21

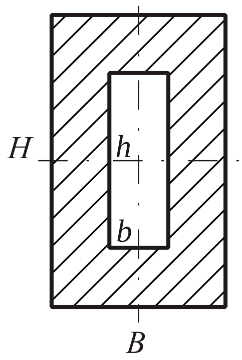


Fig. 22

Note, that allowable value of internal bending moment is determined from condition of strength:

$$[M]\le W_y[\sigma].$$

On the other hand, the method of section connects the external forces with internal bending moments. That is why we begin from calculating the internal forces to find critical cross-section.

Solution

1) Calculating the reactions in supports:

a) $\sum M_A=0:\quad M+R_B\times 2a-q\times 3a\times\dfrac{3a}{2}=0,$

$$R_B\times 2a=\frac{9}{2}qa^2-2qa^2=\frac{5}{2}qa^2\rightarrow R_B=\frac{5}{4}qa;$$

b) $\sum M_B=M+2qa\times a-qa\times\dfrac{a}{2}-R_A\times 2a=0,$

$$R_A \times 2a = 2qa^2 + 2qa^2 - \frac{1}{2}qa^2 \to R_A = \frac{7}{4}qa.$$

c) Checking: $\sum F_z = R_A + R_B - q \times 3a = \frac{5}{4}qa + \frac{7}{4}qa - 3qa = 0$.

The reactions are correct.

2) Designing the graphs of shear forces $Q_z(x)$ and bending moments $M_y(x)$, applying the method of sections.

I–I $0 \le x \le a$

$$Q_z{}^{\mathrm{I}}(x) = qx\big|_{x=0} = 0\big|_{x=a} = qa,$$

$$M_y{}^{\mathrm{I}}(x) = -\frac{qx^2}{2}\Big|_{x=0} = 0\Big|_{x=a} = -\frac{1}{2}qa^2.$$

II–II $0 \le x \le a$

$$Q_z{}^{\mathrm{II}}(x) = q(a+x) - R_B = q(a+x) - \frac{5}{4}qa = qx - \frac{1}{4}qa\Big|_{x=0} = -\frac{1}{4}qa\Big|_{x=2a} = \frac{7}{4}qa.$$

Finding of extremum bending moment value due to intersecting the shear force graph with x-axis in the second portion.

$Q_z(x_0) = qx_0 - \frac{1}{4}qa = 0 \to x_0 = \frac{1}{4}a$ – coordinate of the intrsection.

$$M_y{}^{\mathrm{II}}(x) = -\frac{q(x+a)^2}{2} + R_B x = -\frac{q(x+a)^2}{2} + \frac{5}{4}qax\Big|_{x=0} =$$

$$= -\frac{1}{2}qa^2\Big|_{x=2a} = -2qa^2\Big|_{x=0.25a} = -\frac{15}{32}qa^2.$$

In result, $\left|M_{y\max}\right| = 2qa^2$ (see Fig. 21).

3) Calculating the sectional modulus of the cross-section (see Fig. 22):

$$W_y = \frac{I_1 - I_2}{z_{\max}} = \frac{\frac{BH^3}{12} - \frac{bh^3}{12}}{\frac{H}{2}} = \frac{\frac{6 \times 10^3}{12} - \frac{2 \times 4^3}{12}}{5} = \frac{600 - 2 \times 64}{60} = 98\,\mathrm{cm}^3.$$

4) Calculating the allowable loading $[q]$:

$$\sigma = \frac{M_{y\max}}{W_y} \le [\sigma], \quad M_{y\max} \le [\sigma] W_y$$

$$2[q]a^2 = [\sigma]W_y \rightarrow [q] = \frac{[\sigma]W_y}{2a^2} = \frac{200\times 10^6 \times 98 \times 10^{-6}}{2a^2} = \frac{9800}{a^2} = 9.8\,\text{kN/m}.$$

Example 4 Checking problem for cantilever in plane bending

Given: cast iron cantilever beam of T-section is loaded by the forces: distributed load $q = 10\,\text{kNm}$, concentrated moment $M = 20\,\text{kNm}$ and concentrated force $P = 10\,\text{kN}$. Cross-sectional dimansions: $a = 6\,\text{cm}$, $b = 1\,\text{cm}$, $c = 10\,\text{cm}$, $d = 12\,\text{cm}$. Allowable stresses: $[\sigma]_t = 200\,\text{MPa}$, $[\sigma]_c = 400\,\text{MPa}$, length of the portion $l = 1\,\text{m}$.

R.D.: check the cantilever strength and select optimal orientation of the cross-section relative to plane of loading.

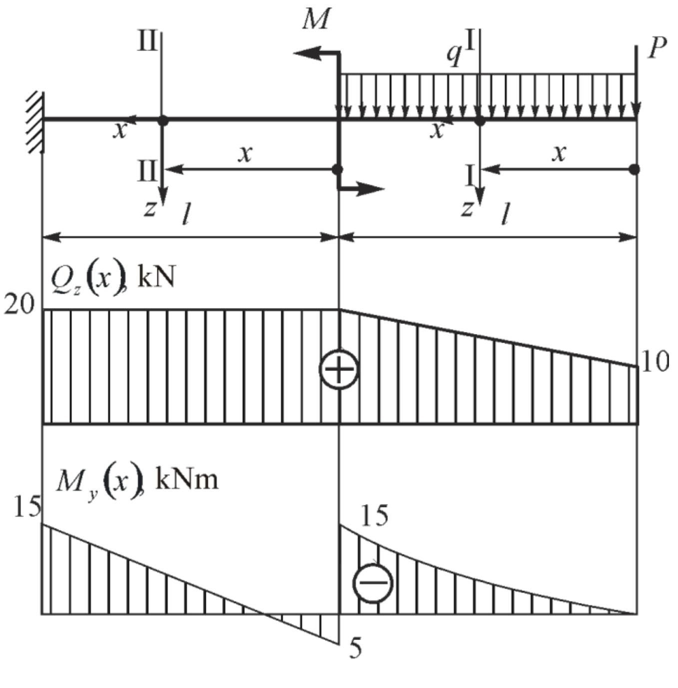


Fig. 23

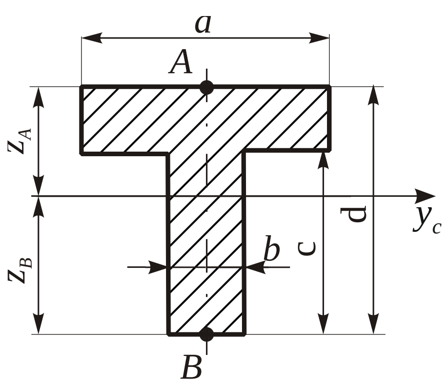


Fig.24

Solution

1) Calculating the internal forces and moments in the cantilever cross-sections

I–I $0 \le x \le 1\,\text{m}$

$$Q_z^{\text{I}}(x) = P + qx\big|_{x=0} = 10\big|_{x=l} = 20\,\text{kN};$$

$$M_y^{\mathrm{I}}(x) = -Px - \frac{qx^2}{2}\Big|_{x=0} = 0\Big|_{x-l} = -15\,\text{kNm}.$$

II–II $0 \le x \le 1\,\text{m}$

$$Q_z^{\mathrm{II}}(x) = P + ql = 20\,\text{kN};$$

$$M_y^{\mathrm{II}}(x) = -P(x+l) - \frac{q(x+l)^2}{2} + M\Big|_{x=0} = 5\Big|_{x=l} = -15\,\text{kNm}.$$

2) Calculating the neutral axis position. For this, let us select *y*-axis as original (see Fig. 25).

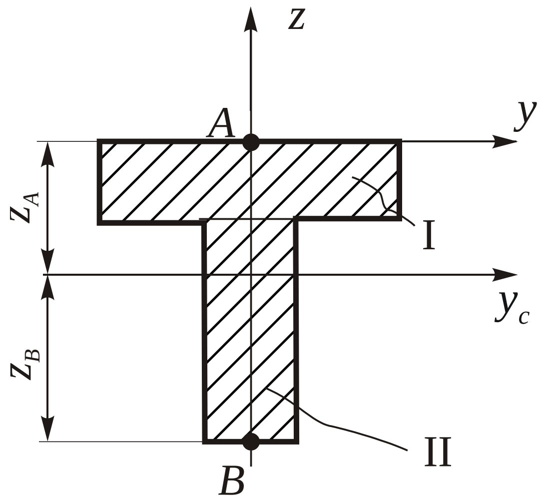


Fig. 25

a) Determine first moment of cross-sectional area relative to *y*-axis:

$$S_y^{\mathrm{I}} = a(d-c)\frac{d-c}{2} =$$

$$= 6\times(12-10)\times\frac{12-10}{2} = 12\,\text{cm}^3;$$

$$S_y^{\mathrm{II}} = bc\left(d - \frac{c}{2}\right) =$$

$$= 1\times 10\times\left(12 - \frac{10}{2}\right) = 70\,\text{cm}^3.$$

b) determine the neutral axis position using the formula

Vertical coordinate

$$z_c = \frac{S_y^{\mathrm{I}} + S_y^{\mathrm{II}}}{A^{\mathrm{I}} + A^{\mathrm{II}}} = \frac{12+70}{6\times(12-10)+1\times 10} = 3.73\,\text{cm} \text{ and } z_A = 3.73\,\text{cm},\ z_B = 8.27\,\text{cm}.$$

3) Calculating the central moment of inertia.

$$I_{y_c}^{\mathrm{I}} = \frac{(d-c)a^3}{12} + (d-c)a\left(z_A - \frac{d-c}{2}\right)^2 =$$

$$= \frac{(12-10)\times 6^3}{12} + (12-10)\times 6\times\left(3.73 - \frac{12-10}{2}\right)^2 = 125.44\,\text{cm}^4;$$

$$I_{y_c}{}^{II} = \frac{b(d-c)^3}{12} + b(d-c)\left(d - \frac{c}{2} - z_A\right)^2 =$$

$$= \frac{1\times(12-10)^3}{12} + 1\times(12-10)\times\left(12 + \frac{10}{2} - 3.73\right)^2 = 278.68\,\text{cm}^4.$$

$$I_{y_c} = I_{y_c}{}^{\text{I}} + I_{y_c}{}^{\text{II}} = 125.44 + 278.68 = 404.12\,\text{cm}^4.$$

4) Calculating the maximum normal stresses in A and B points (see Fig. 26):

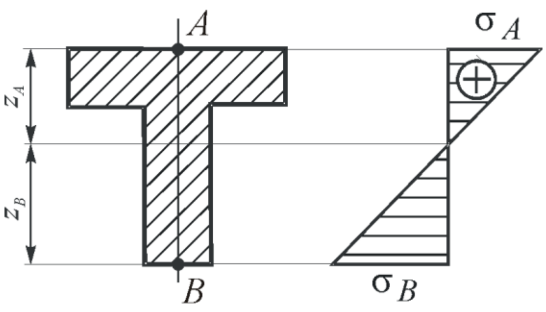


Fig. 26

$$\sigma_A = \frac{M_{y\max} z_A}{I_{y_c}} =$$

$$= \frac{15\times10^3 \times 3.73\times10^{-2}}{404.12\times10^{-8}} = 139.1 \text{ MPa},$$

$$\sigma_B = \frac{M_{y\max} z_B}{I_{y_c}} =$$

$$\frac{15\times10^3 \times (12-3.73)\times10^{-2}}{404.12\times10^{-8}} = 308.5\,\text{MPa}.$$

5) Checking the strength of the beam in two possible cases of its orientation to determine its optimal orientation relative to the plane of loading.

Let us compare stresses in A and B points for two possible cases of cross-section orientation.

For orientation I (see Fig. 27)

$\sigma_A = \sigma_{\max}^{tens} = 139.1\,\text{MPa}$. Since $[\sigma]_t = 200\,\text{MPa}$ the most tensile point A is strong.

Simultaneously,

$\sigma_B = \sigma_{\max}^{comp} = 308.5\,\text{MPa}$. Since $[\sigma]_c = 400\,\text{MPa}$ the most compressed point B is also strong. Therefore, this orientation of cross-section corresponds to the condition of strength (see Fig. 27).

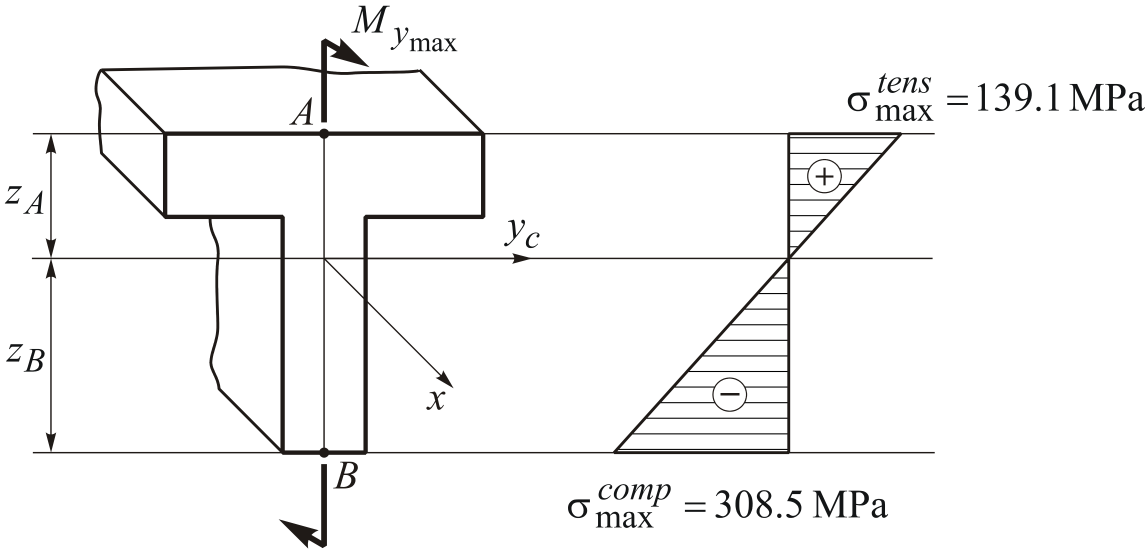


Fig. 27

For orientation II

$\sigma_A = \sigma_{\max}^{comp} = 139.1$ MPa. Since $[\sigma]_c = 400$ MPa the most compressed point A is strong.

Simultaneously,

$\sigma_B = \sigma_{\max}^{tens} = 308.5$ MPa. Since $[\sigma]_c = 200$ MPa the most tensile point B becomes unstrong (Fig. 28).

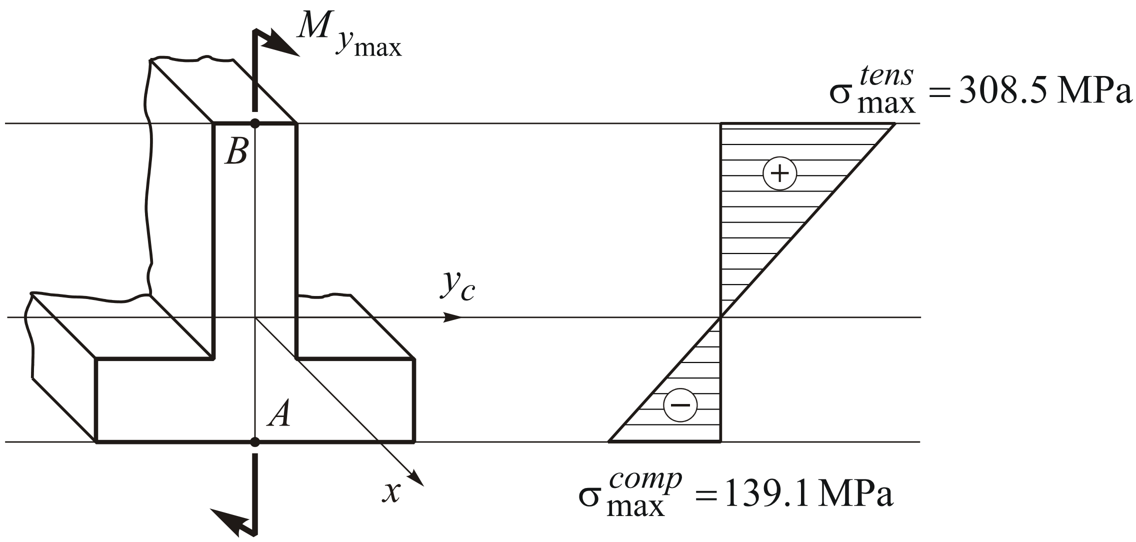


Fig. 28

In result, this orientation of cross-section does not satisfy the condition of strength.

Conclusion. To be situated rationally, larger acting stresses must correspond to larger allowable stresses.

Example 4 Design problem for round simple beam

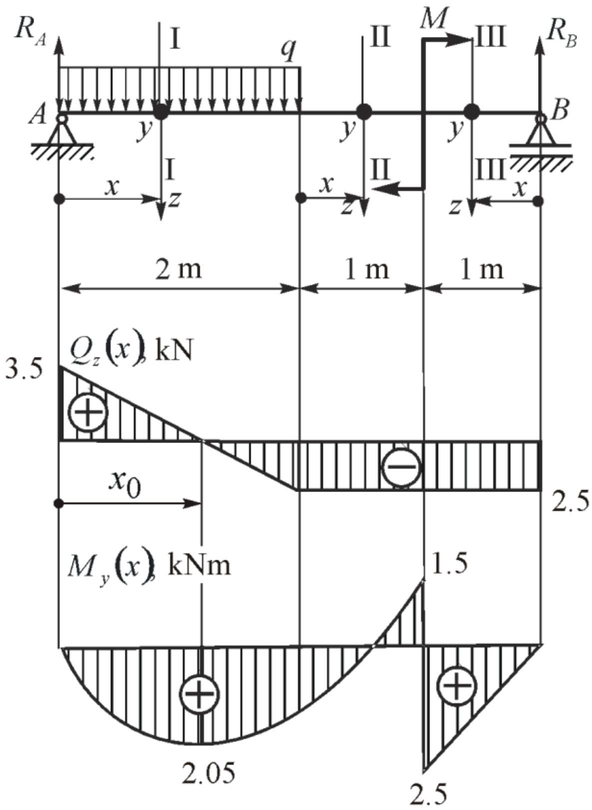


Fig. 29

Given: $M = 4\,\text{kNm}$, $q = 3\,\text{kN/m}$, $[\sigma] = 160\,\text{MPa}$.

R.D.: Diameter of round cross-section.

Solution

1) Calculating the support reactions:

(a) $\sum M_B = 0$:

$$-R_A \times 4 + q \times 2 \times 3 - M = 0;$$

$$R_A = \frac{3 \times 2 \times 3 - 4}{4} = 3.5\,\text{kN};$$

(b) $\sum M_A = 0$:

$$R_B \times 4 - M - q \times 2 \times 1 = 0;$$

$$R_B = \frac{4 + 3 \times 2 \times 1}{4} = 2.5\,\text{kN}.$$

(c) Checking: $\sum F_z = 0$:

$$3.5 + 2.5 - 3 \times 2 = 0.$$

The reactions are correct.

2) Designing the graphs of shear forces $Q_z(x)$ and bending moments $M_y(x)$, applying the method of sections.

I–I $0 \le x \le 2\,\text{m}$

$$Q_z^{\text{I}}(x) = R_A - qx = 3.5 - 3x\big|_{x=0} = 3.5\big|_{x=2} = -2.5\,\text{kN}.$$

In cross-section with $Q_z = 0$, M_y graph grows to maximum value.

Using equation $Q_z^{\text{I}}(x_0) = 0$ let us find x_0:

$$3.5 - 3x_0 = 0 \to x_0 = 1.17\,\text{m}.$$

$$M_y^{\text{I}}(x) = R_A x - q\frac{x^2}{2} = 3.5x - 1.5x^2\big|_{x=0} = 0\big|_{x=1.17} = 2.05\big|_{x=2} = 1\,\text{kNm}.$$

II–II $0 \le x \le 1\,\text{m}$

$$Q_z^{\text{II}} = R_A - q \times 2 = 3.5 - 3 \times 2 = -2.5\,\text{kN}.$$

$$M_y^{\text{II}}(x) = R_A \times (x+2) - q \times 2 \times (x+1) =$$

$$= 3.5 \times (x+2) - 6 \times (x+1)\big|_{x=0} = 1\big|_{x=1} = -1.5\,\text{kNm}.$$

III–III $0 \le x \le 1\,\text{m}$

$$Q_z^{\text{III}}(x) = -R_B = -2.5\,\text{kN}.$$

$$M_y^{\text{III}}(x) = R_B x = 2.5x\big|_{x=0} = 0\big|_{x=1} = 2.5\,\text{kNm}.$$

Conclusion. $\left|M_{y\max}\right| = 2.5\,\text{kNm}.$

3) Determining the diameter of round beam from the condition of strength in critical cross-section with $\left|M_{y\max}\right| = 2.5\,\text{kNm}.$

$$\sigma_{\max} = \frac{M_{y\max}}{W_y} \le [\sigma] \to W_y \ge \frac{M_{y\max}}{[\sigma]} = \frac{2.5 \times 10^3}{160 \times 10^6} = 15.6 \times 10^{-6}\,\text{m}^3 = 15.6\ \text{cm}^3.$$

$$W_y = \frac{\pi d^3}{32} \to d = \sqrt[3]{\frac{32 W_y}{\pi}} = \sqrt[3]{\frac{32 \times 15.6 \times 10^{-6}}{\pi}} = 5.4 \times 10^{-2}\,\text{m}.$$

Example 5 Checking problem for I-beam

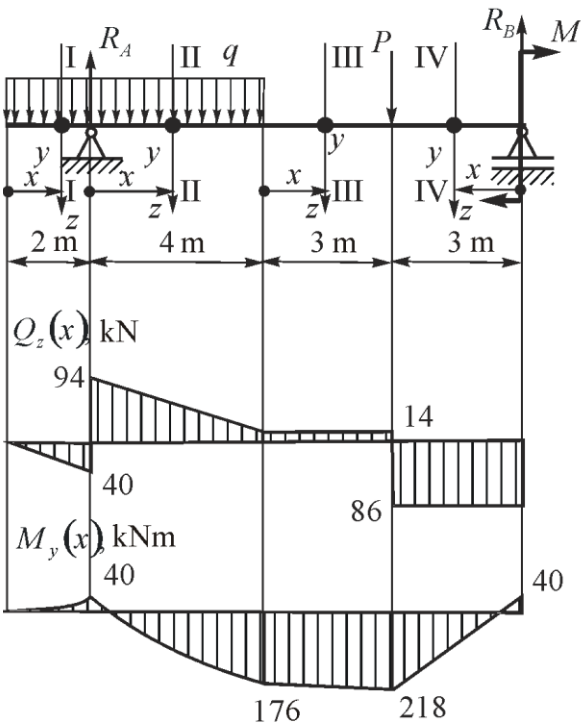


Fig. 30

Given: Cross-section I-beam №50, $[\sigma] = 160\,\text{MPa}$, $P = 100\,\text{kN}$, $M = 40\,\text{kNm}$, $q = 20\,\text{kN/m}$.

R.D.: Check the strength of the beam.

Solution

1) Calculating the support reactions:

(a) $\sum M_B = 0$:

$$-R_A \times l + q \times 6 \times 9 + P \times 3 - M = 0;$$

$$R_A = \frac{20 \times 6 \times 9 + 100 \times 3 - 40}{10} = 134\,\text{kN}.$$

(b) $\sum M_A = 0$:

$$R_B \times l - P \times 7 - M - q \times 6 \times 1 = 0;$$

$$R_B = \frac{100\times 7 + 40 + 20\times 6\times 1}{10} = 86\,\text{kN}.$$

(c) Checking: $\sum F_z = 0$:

$$134 + 86 - 20\times 6 - 100 = 0.$$

The reactions are correct.

2) Calculating the shear forces $Q_z(x)$ and bending moments $M_y(x)$ applying the method of sections.

I–I $0 \le x \le 2\,\text{m}$

$$Q_z^{\text{I}}(x) = -qx = -20x\big|_{x=0} = 0\big|_{x=2} = -40\,\text{kN};$$

$$M_y^{\text{I}}(x) = -\frac{qx^2}{2} = -10x^2\big|_{x=0} = 0\big|_{x=2} = 40\,\text{kNm};$$

II–II $0 \le x \le 4\,\text{m}$

$$Q_z^{\text{II}}(x) = -q\left(x+2\right) + R_A = -20\left(x+2\right) + 134\big|_{x=0} = 94\big|_{x=4} = 14\,\text{kN};$$

$$M_y^{\text{II}}(x) = -\frac{q\left(x+2\right)^2}{2} + R_A x = -10(x+2)^2 + 134x\big|_{x=0} = -40\big|_{x=4} = 176\,\text{kNm};$$

III–III $0 \le x \le 3\,\text{m}$

$$Q_z^{\text{III}}(x) = -q\times 6 + R_A = -20\times 6 + 134 = 14\,\text{kN};$$

$$M_y^{\text{III}}(x) = -6q(x+3) + R_A(x+4) =$$

$$= -120(x+3) + 134(x+4)\big|_{x=0} = 176\big|_{x=3} = 218\,\text{kNm};$$

IV–IV $0 \le x \le 3\,\text{m}$

$$Q_z^{\text{IV}}(x) = -R_B = -86\,\text{kN};$$

$$M_y^{\text{IV}}(x) = -M + R_B x = -40 + 86x\big|_{x=0} = -40\big|_{x=3} = -218\,\text{kNm}.$$

The graphs are shown in Fig. 30.

3) Checking the strength of I-beam №50. From assortment (GOST 8239-72) find its sectional modulus $W_y = 1589\,\text{cm}^3$. In our case, $\left|M_{y\max}\right| = 218\,\text{kNm}$.

Condition of strength is

$$\sigma_{\max} = \frac{\left|M_{y\max}\right|}{W_y} = \frac{218\times10^{-3}}{1589\times10^{-6}} = 137.2\,\text{MPa} < [\sigma] = 160\,\text{MPa}.$$

Conclusion. The beam is strong.

Example 6 Checking problem for cast iron beam

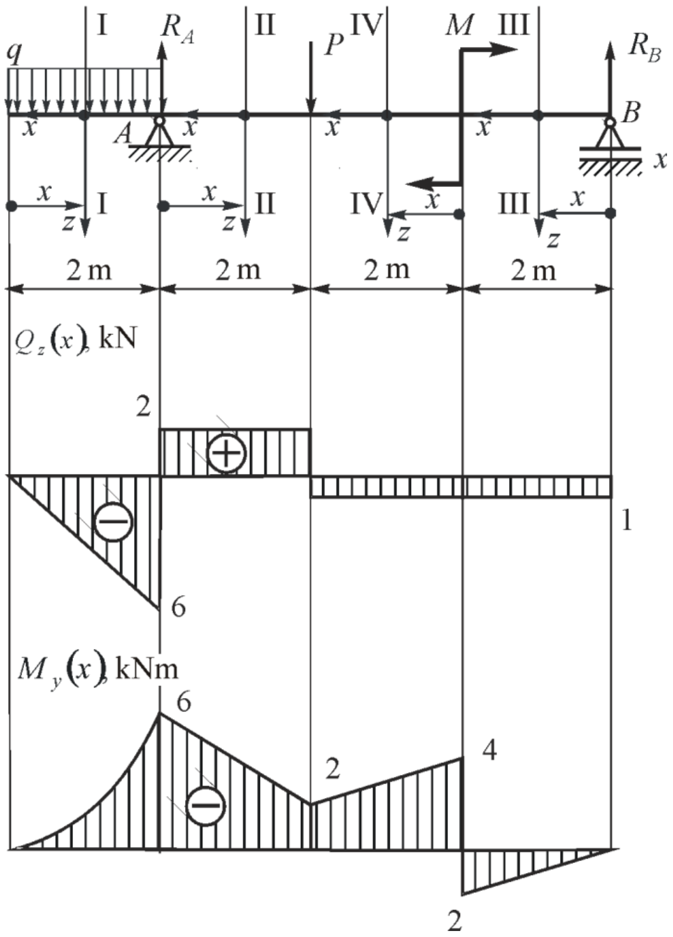


Fig. 31

Given: cast iron of T-section is loaded by $q = 3\,\text{kN/m}$, $P = 3\,\text{kN}$, $M = 6\,\text{kNm}$. Cross-sectional dimensions are shown in Figs. 32 and 33 for two possible orientations of the section. Allowable stresses for cast iron are $[\sigma]_t = 40\,\text{MPa}$, $[\sigma]_c = 80\,\text{MPa}$.

R.D.: find optimal orientation of the beam relative to the plane of loading from two possible versions shown in Figs. 32 and 33 check its strength.

Solution

1) Calculating the support reactions.

(a) $\sum M_B = 0$:

$$-R_A\times6 + q\times2\times7 + P\times4 - M = 0;$$

$$R_A = \frac{3\times2\times7+3\times4-6}{6} = 8\,\text{kN};$$

(b) $\sum M_A = 0$:

$$R_B\times6 - P\times2 + q\times2\times1 - M = 0; \quad R_B = \frac{6+3\times2-3\times2\times1}{6} = 1\,\text{kN}.$$

(c) Checking: $\sum F_z = 0$:

$$-3\times 2+8-3+1=0. \quad \text{Reactions are correct.}$$

2) Calculating the shear forces and bending moments in cross-sections of the beam.

I–I $0 \le x \le 2\,\text{m}$

$$Q_z^{\text{I}}(x) = -qx = -3x\big|_{x=0} = 0\big|_{x=3} = -6\text{kN};$$

$$M_y^{\text{I}}(x) = -\frac{qx^2}{2} = -1.5x^2\big|_{x=0} = 0\big|_{x=2} = -6\,\text{kNm};$$

II–II $0 \le x \le 2\,\text{m}$

$$Q_z^{\text{II}}(x) = -2q + R_A = -2\times 3+8 = 2\,\text{kN};$$

$$M_y^{\text{II}}(x) = -2q(x+1) + R_A x = -6(x+1)+8x\big|_{x=0} = -6\big|_{x=2} = -2\,\text{kNm};$$

III–III $0 \le x \le 2\,\text{m}$

$$Q_z^{\text{III}}(x) = -R_B = -1\text{kN};$$

$$M_y^{\text{III}}(x) = R_B x = x\big|_{x=0} = 0\big|_{x=2} = 2\,\text{kNm};$$

IV–IV $0 \le x \le 2\,\text{m}$

$$Q_z^{\text{IV}}(x) = -R_B = -1\text{kN};$$

$$M_y^{\text{IV}}(x) = R_B(x+2) - M = x+2-6\big|_{x=0} = 4\big|_{x=2} = 2\,\text{kNm}.$$

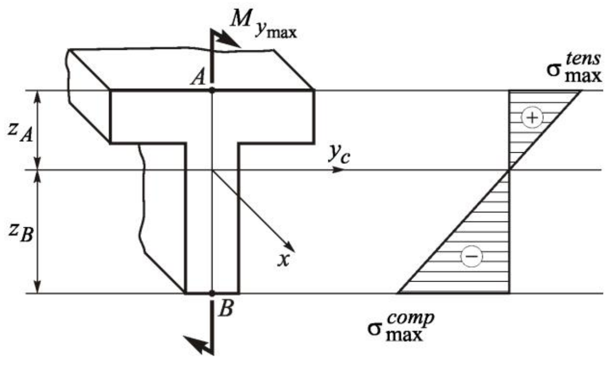


Fig. 32

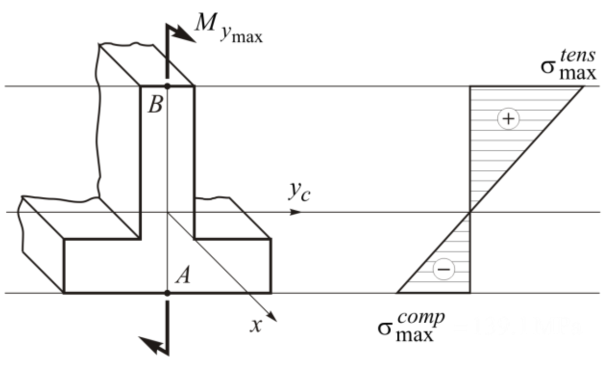


Fig. 33

3) Selection of cross-section optimal orientation in plane bending comparing two orientations shown in Figs. 32 and 33.

Note, that the T-section is non-symmetrical relative to neutral axis. Largest stresses appear in the most remote points of the section (points *A* and *B*). Since for cast iron $[\sigma]_t < [\sigma]_c$, the largest compressed stresses must appear in the layers situated at the larger distance from neutral axis i.e. at z_B distance. That is why the orientation shown in Fig. 32 is optimal in this case of loading.

4) Checking the strength of the cross-section shown in Fig. 32 (of optimal orientation).

Let us write conditions the strength for *A* and *B* points:

$$\sigma_B = \sigma_{\max}^{comp} = \frac{\left|M_{y\max}\right| z_B}{I_{y_c}} \le [\sigma]_c;$$

$$\sigma_A = \sigma_{\max}^{tens} = \frac{\left|M_{y\max}\right| z_A}{I_{y_c}} \le [\sigma]_t.$$

For this, let us calculate central moments of inertia I_{y_c} calculating preliminary vertical coordinate of the cross-sectional centroid. We will use y_0-axis as the origin (see Fig. 34).

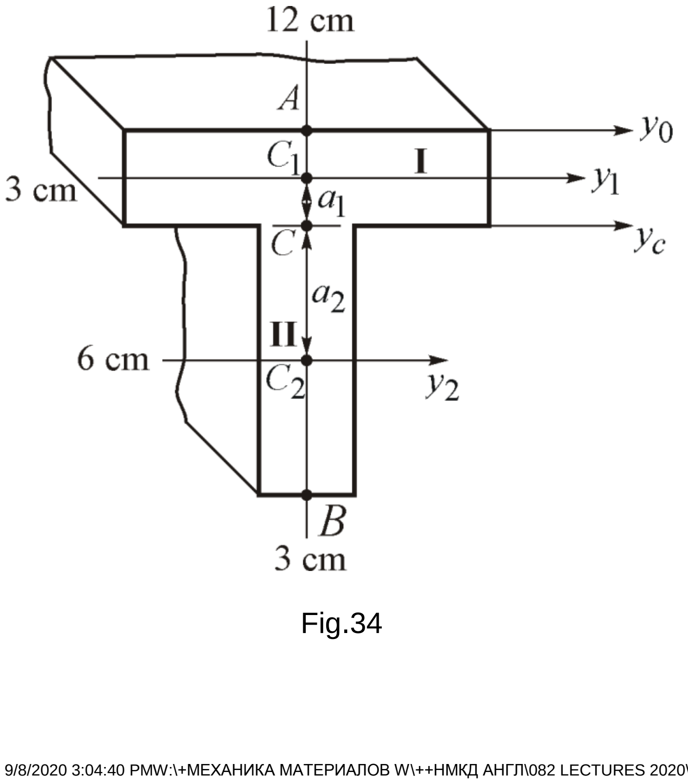


Fig.34

Coordinate of centroid is

$$z_c = \frac{S_{y_0}}{A} = \frac{A_1 z_1 + A_2 z_2}{A_1 + A_2} = $$
$$= \frac{12\times 3\times 1.5 + 3\times 6\times 6}{12\times 3 + 3\times 6} = 3\,\text{cm}.$$

Central moment of inertia is:

$$I_{y_c} = I_{y_1}{}^{\text{I}} + a_1{}^1 A_1 + I_{y_2}{}^{\text{II}} + a_2{}^2 A_2 =$$
$$\frac{12\times 3^3}{12} + (-1.5)^2\times 36 + \frac{3\times 6^3}{12} + 3^2\times 18 =$$
$$= 324\,\text{cm}^4.$$

Then

$$\sigma_A = \sigma_{\max}^{tens} = \frac{6\times10^3\times3\times10^{-2}}{324\times10^{-8}} = 35.5\ \text{MPa} < [\sigma]_t = 40\ \text{MPa};$$

$$\sigma_B = \sigma_{\max}^{comp} = \frac{6\times10^3\times6\times10^{-2}}{324\times10^{-8}} = 71\ \text{MPa} < [\sigma]_c = 80\ \text{MPa}.$$

Conclusion. Since both condition of strength are satisfied the beam is strong.

Example 7 Problem of allowable external load

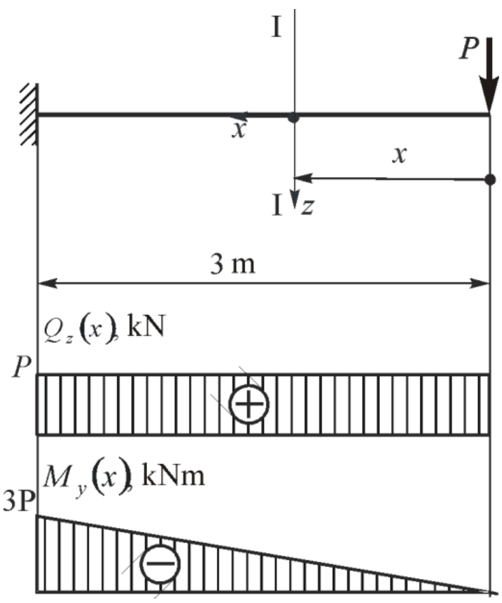


Fig. 35

Given: steel I-beam No24 cantilever is loaded by the force *P*. The length $l = 3\,\text{m}$, $[\sigma] = 160\,\text{MPa}$.

R.D.: calculate allowable value if the *P*-force from condition of strength.

Solution

1) Calculating the internal forces in the cantilever applying the method of sections.

I–I $0 \le x \le 2\,\text{m}$

$$Q_z^{\text{I}}(x) = P;$$

$$M_y^{\text{I}}(x) = -Px\big|_{x=0} = 0\big|_{x=3} = -3P.$$

Corresponding graphs are shown in Fig. 35.

2) Calculating the allowable force value [*P*].

Critical section is situated in rigid support and $|M_{y\max}| = 3P$.

From condition of strength:

$$\sigma_{\max} = \frac{[M_y]}{W_y} = [\sigma] \to [M_y] = [\sigma]W_y.$$

From assortment, (GOST 8239-72) for I-beam №24 we find its sectional modulus $W_y = 289\,\text{cm}^3$.

After this, allowable internal bending moment becomes

$[M_y] = 160\times10^6\times289\times10^{-6} = 46.24\,\text{kNm}$ and allowable force is

$$[P]=\frac{[M_y]}{3}=\frac{46.24\times10^3}{3}=15.4\,\text{kN}.$$

Example 8 Problem of optimal supports placement in plane bending

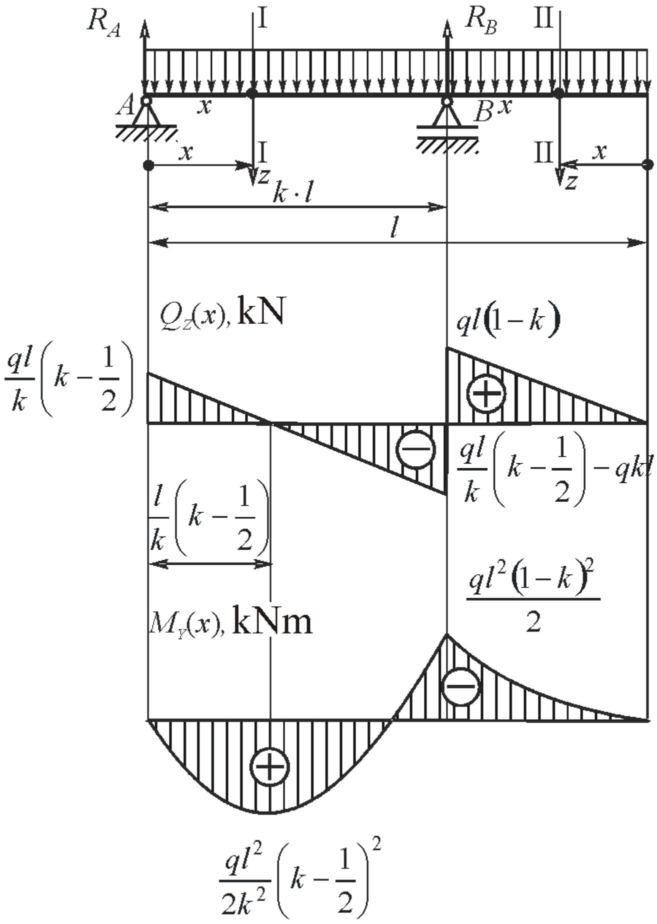


Fig. 36

Given: simple beam with overhang is loaded by uniformly distributed load q. Position of the right support is determined by the k-factor.

R.D.: Calculate the k-factor value which provides the largest allowable value $[q]$.

Solution

1) Calculating the support reactions.

(a) $\sum M_A=0$:

$$\frac{ql^2}{2}-R_B\times kl=0;\quad R_B=\frac{ql}{2k}.$$

(b) $\sum M_B=0$:

$$ql\left(kl-\frac{l}{2}\right)-R_A\times kl=0;$$

$$R_A=\frac{ql}{k}\left(k-\frac{1}{2}\right).$$

(c) Checking: $\sum F_z=ql-\frac{ql}{2k}-\frac{ql}{k}\left(k-\frac{1}{2}\right)=ql-\frac{ql-2kql+ql}{2k}=0$.

The reactions are correct.

2) Calculating the shear forces $Q_z(x)$ and bending moments $M_y(x)$ applying the method of sections.

I–I $0\le x\le kl$

$$Q_z^{\mathrm{I}}(x) = R_A - qx = \frac{ql}{k}\left(k - \frac{1}{2}\right) - qx\Big|_{x=0} = \frac{ql}{k}\left(k - \frac{1}{2}\right)\Big|_{x=kl} = \frac{ql}{k}\left(k - \frac{1}{2}\right) - qkl\,.$$

In this portion, M_y graph has maximum value which is calculated equating shear force to 0:

$$Q_z^{\mathrm{I}}(x_0) = 0; \quad \frac{ql}{k}\left(k - \frac{1}{2}\right) - qx_0 = 0 \rightarrow x_0 = \frac{l}{k}\left(k - \frac{1}{2}\right).$$

$$M_y^{\mathrm{I}}(x) = R_A x - \frac{qx^2}{2}\Big|_{x=0} = 0\Big|_{x=kl} = \frac{ql^2(1-k)^2}{2}\Big|_{x=x_0} = \frac{ql^2}{2k}\left(k - \frac{1}{2}\right)^2.$$

II–II $0 \le x \le (1-k)l$

$$Q_z^{\mathrm{II}}(x) = qx\Big|_{x=0} = 0\Big|_{x=l(1-k)} = ql(1-k);$$

$$M_y^{\mathrm{II}}(x) = \frac{qx^2}{2}\Big|_{x=0} = 0\Big|_{x=l(1-k)} = \frac{ql^2(1-k)^2}{2}.$$

3) Calculating the *k*-factor.

It is evident that [*q*] will be the largest in value if maximum bending moments in both portions will be equal to teach other:

$$\left|M_y^{\mathrm{I}}\right|_{\max} = \left|M_y^{\mathrm{II}}\right|_{\max};$$

$$\frac{ql^2}{2k^2}\left(k - \frac{1}{2}\right)^2 = \frac{ql^2(1-k)^2}{2} \rightarrow \left(k - \frac{1}{2}\right)^2 = k^2(1-k)^2 \rightarrow$$

$$\begin{cases} k - \frac{1}{2} = k(1-k); \\ k - \frac{1}{2} = k(k-1); \end{cases} \rightarrow \begin{cases} k^2 = \frac{1}{2}; \\ k^2 - 2k + \frac{1}{2} = 0; \end{cases}$$

$$k_{1,2} = \pm\frac{1}{\sqrt{2}}; \quad k_{3,4} = 1 \pm \frac{1}{\sqrt{2}}.$$

Therefore, $k = \frac{1}{\sqrt{2}}$ or $k = 1 - \frac{1}{\sqrt{2}}$, since another values of *k* contradict the problem data.

Example 9 Design problem for cantilever

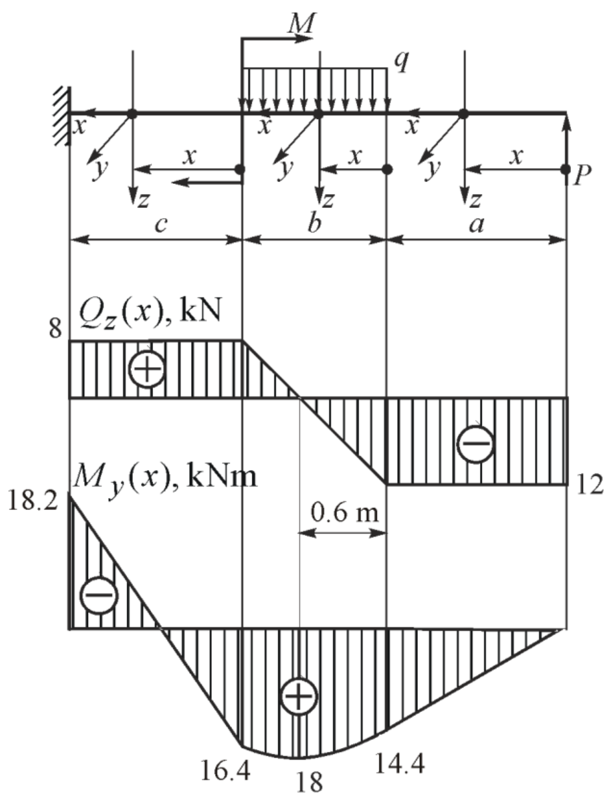


Fig. 37

Given: steel cantilever is loaded by the following loadings: $q = 20$ kN/m, $P = 12$ kN, $M = 25$ kNm. The cantilever lengths: $a = 1.2$ m, $b = 1$m, $c = 1.2$ m. $[\sigma] = 200$ MPa.

R.D.: 1) diameter of solid round section; 2) dimensions of rectangle cross-section; 3) the number of I-beam section.

Solution

1) Calculating the shear forces $Q_z(x)$ and bending moments $M_y(x)$, applying the method of sections.

I–I $0 \le x \le 1.2$ m

$$Q_z^{\text{I}}(x) = -P = -12\,\text{kN};$$

$$M_y^{\text{I}}(x) = Px = 12x\big|_{x=0} = 0\big|_{x=1.2} = 14.4\,\text{kNm};$$

II–II $0 \le x \le 1$m

$$Q_z^{\text{II}}(x) = -P + qx = -12 + 20x\big|_{x=0} = -12\big|_{x=1} = 8\,\text{kNm};$$

$$Q_z^{\text{II}}(x_0) = -12 + 20x_0 = 0 \to x_0 = \frac{12}{20} = 0.6\,\text{m};$$

$$M_y^{\text{II}}(x) = P(x+a) - \frac{qx^2}{2} = 12(x+1.2) - 10x^2\big|_{x=0} = 14.4\big|_{x=0.6} = 18\big|_{x=1} = 16.4\,\text{kNm};$$

III–III $0 \le x \le 1{,}2$ m.

$$Q_z^{III}(x) = -P + qb = 8\,\text{kN};$$

$$M_y^{III}(x) = P\left(x+a+b\right) - qb\left(x+\frac{b}{2}\right) - M = 12\times(x+2.2) - 20\times(x+0.5) - 25\Big|_{x=0} =$$

$$= 14{,}4\Big|_{x=1.2} = 18.2\,\text{kNm}.$$

Conclusion. Critical cross-section is determined taking into account maximum value of the bending moment:

$\left|M_y\right|_{\max} = 18.2\,\text{kNm}$. Corresponding shear force $Q_z = 8\,\text{kN}$.

2) Calculating the sectional modulus of the cross-section using condition of strength.

$$\sigma_{\max} = \frac{\left|M_y\right|_{\max}}{W_y} \le [\sigma]; \quad W_y \ge \frac{\left|M_y\right|_{\max}}{[\sigma]} = \frac{18.2\times10^3}{200\times10^6} = 91\times10^{-6}\,\text{m}^3.$$

3) Knowing $W_y \ge 91\cdot10^{-6}\,\text{m}^3$ let us find corresponding I-beam number from assortment. Closest less I-beam section is №14 with $W_y = 81.7\times10^{-6}\,\text{m}^3$. It will be evidently overstressed:

$$\sigma_{\max} = \frac{\left|M_y\right|_{\max}}{W_y} = \frac{18.2\times10^3}{81.7\times10^{-6}} = 222.8\,\text{MPa}.$$

Overstress is $\Delta = \frac{222.8-200}{200}\times100\% = 11.4\% > 5\%$. Since overstress is non-permissible we will find in assortment larger I-beam section №16 with $W_y = 109\times10^{-6}\,\text{m}^3$. Its understress is

$$\sigma_{\max} = \frac{\left|M_y\right|_{\max}}{W_y} = \frac{18.2\times10^3}{109\times10^{-6}} = 167\,\text{MPa}.$$

Understress $\Delta = \frac{200-167}{200}\times100\% = 16.5\%$. The dimensions of selected I-beam are shown in the Table:

h	b	d	t	A cm^2	I_y cm^4	W_y cm^3	S_y cm^3
mm							
160	81	5.0	7.8	20.2	873	109	62.3

Drawing the graphs of acting stress distribution $\sigma_x(z)$ and $\tau_x(z)$ in critical cross-section.

Calculating the shear stresses in specific points of the section: pp. 1, 2^f, 2^w, 3. Note, that 2^f point belongs to the flange and 2^w point belongs to the web.

General Juravsky formula is

$$\tau(z) = \frac{Q_z S_y{}^*}{bI_y}.$$

It is evident that $\tau_3 = 0$ since $S_y{}^* = 0$.

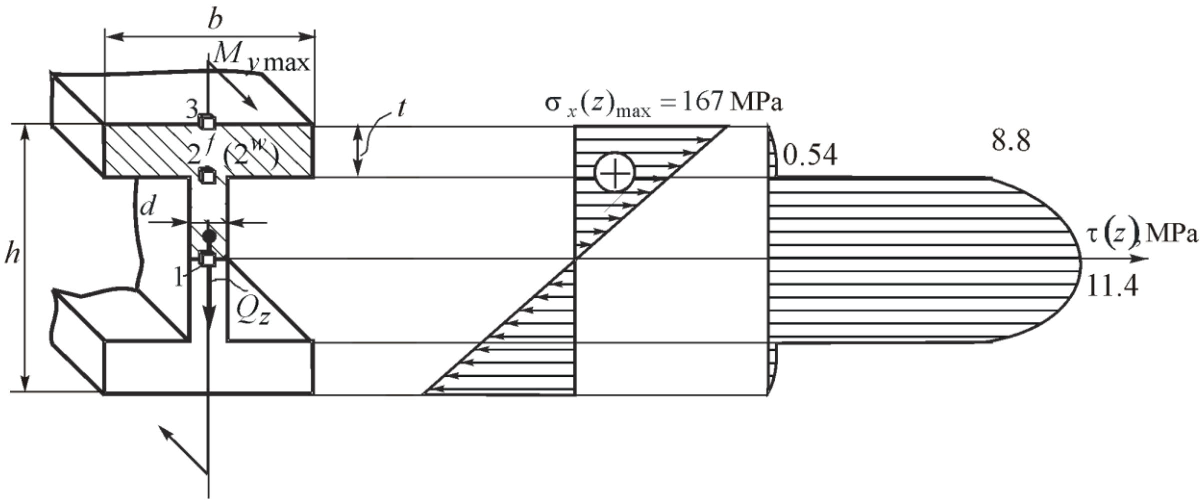


Fig. 38

For 2^f-point the formula becomes:

$$\tau_{p.2^f} = \frac{Q_z bt\left(\frac{h}{2}-\frac{t}{2}\right)}{bI_y} = \frac{8\times10^3\times7.8\times10^{-3}\times\left(\frac{160}{2}-\frac{7.8}{2}\right)\times10^{-3}}{873\times10^{-8}} = 0.54\,\text{MPa}.$$

For 2^w-point the formula is the following:

$$\tau_{p.2^w} = \frac{Q_z bt\left(\frac{h}{2}-\frac{t}{2}\right)}{dI_y} = \frac{8\times10^3\times81\times10^{-3}\times7.8\times10^{-3}\left(\frac{160}{2}-\frac{7.8}{2}\right)\times10^{-3}}{5\times10^{-3}\times873\times10^{-8}} = 8.8\,\text{MPa}.$$

Maximum shear stresses act at the points of neutral axis:

$$\tau_{p.1} = \tau_{\max} = \frac{Q_z S_y{}^T}{dI_y} = \frac{8\times10^3\times62.3\times10^{-6}}{5\times10^{-3}\times873\times10^{-8}} = 11.4\,\text{MPa}.$$

Estimating the stress state of cross-section specific points and their strength. Note, that I-beam section is an example of thin-walled section, that is why we will estimate the stress state of specific points taking into account not only acting normal stresses but also shear ones.

Point 3.

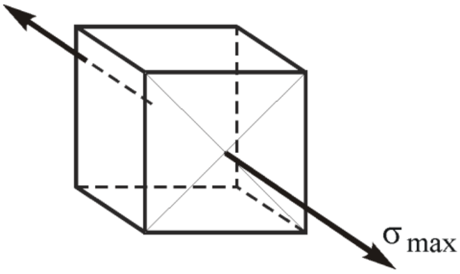


Fig. 39

$$\sigma_{max} = 167\,\text{MPa},\ \tau = 0.$$

Deformation – tension.
Stress-state – uniaxial.
Condition of strength is

$$\sigma_{max} \le [\sigma], \quad 167\text{ MPa} < 200\text{ MPa}.$$

Conclusion: point is strong.

Point 1

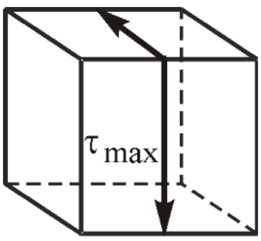


Fig. 40

$$\tau_{max} = 11.4\,\text{MPa}, \quad \sigma = 0.$$

Deformation – pure shear,
Stress state – biaxial. To prove this concept, let us calculate principal stresses:

$$\sigma_{1(3)} = \frac{\sigma_\alpha + \sigma_\beta}{2} \pm \frac{1}{2}\sqrt{(\sigma_\alpha - \sigma_\beta)^2 + 4\tau^2}; \quad \sigma_1 = \tau_{max};$$

$$\sigma_2 = -\tau_{max}.$$

Condition of strength is written applying third strength theory:

$$\sigma_{eq}{}^{III} = |\sigma_1 - \sigma_3| \le [\sigma], \quad \sigma_{eq}{}^{III} = |\tau_{max} + \tau_{max}| \le [\sigma], \quad 22.8\text{ MPa} < 200\text{ MPa}.$$

Conclusion: point is strong.

Point 2^{w}

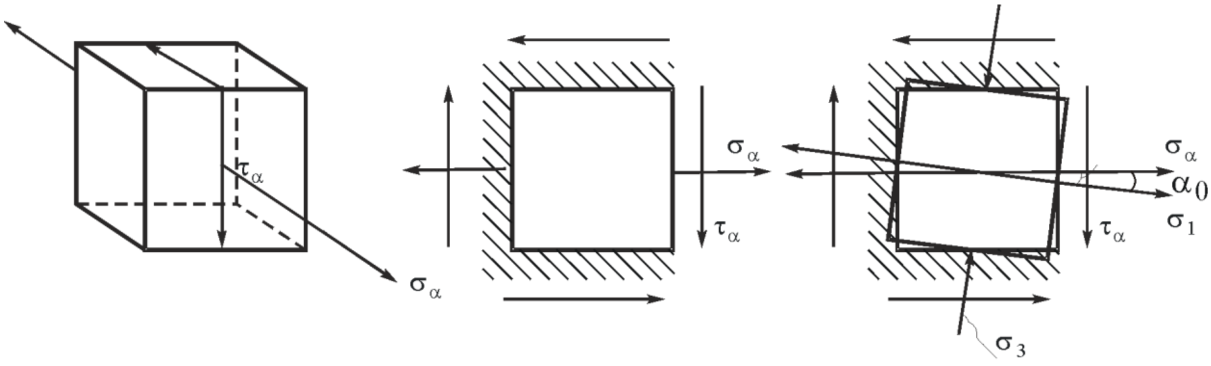


Fig. 41

$$\sigma_a = \sigma_{\text{p.2}^{\text{w}}} = \frac{M_y\left(\frac{h}{2}-t\right)}{I_y} = \frac{M_y\left(\frac{h}{2}-t\right)}{I_y}\times\frac{\frac{h}{2}}{\frac{h}{2}} = \frac{M_y\left(\frac{h}{2}-t\right)}{W_y\frac{h}{2}} = \sigma_{\max}\frac{\left(\frac{h}{2}-t\right)}{\frac{h}{2}} =$$

$$=167\times\frac{80-7.8}{80}=139.2\,\text{MPa}.$$

$$\tau_a = \tau_{\text{p.2}^{\text{w}}} = 8.8\,\text{MPa}.$$

Let us calculate the principal stresses if $\sigma_\alpha = 139.2\,\text{MPa}$, $\sigma_B = 0$, $\tau_\alpha = 8.8\,\text{MPa}$:

$$\sigma_{1(3)} = \frac{\sigma_\alpha+\sigma_\beta}{2}\pm\frac{1}{2}\sqrt{\left(\sigma_\alpha-\sigma_\beta\right)^2+4\tau_\alpha{}^2} = \frac{139.2\times10^6}{2}\pm\frac{1}{2}\times140.3\times10^6 =$$

$$=69.6\text{MPa}\pm70.15\text{MPa}.$$

In this, $\sigma_1 = 139.75\,\text{MPa}$, $\sigma_2 = 0$, $\sigma_3 = -0.55\,\text{MPa}$.

Principal planes are situated at the α_0 angle:

$$\text{tg}2\alpha_0 = \frac{2\tau_\alpha}{\sigma_\beta-\sigma_\alpha} = \frac{2\times8.8}{0-139.2} = -0.126, \quad \alpha_0 = -3°36" \text{ (clockwise rotation).}$$

Conclusion. Stress state at this point is two-dimensional (plane).

4) Calculating the dimensions of rectangle cross-section using condition of strength.

$$W_y = \frac{bh^2}{6} \ge 91\times10^{-6}\,\text{m}^3.$$

In $\frac{h}{b}=2$ we have that $\frac{4}{6}b^3 \ge 91\times10^{-6}$ and $b \ge \sqrt[3]{\frac{3\times91\times10^{-6}}{2}} = 5.15\times10^{-2}\,\text{m}$.

$$h = 2b = 10.3\times10^{-2}\,\text{m}.$$

$$A = hb = 5.15\times10^{-2}\times10.3\times10^{-2} = 53.05\times10^{-4}\,\text{m}^2.$$

Let us draw the graphs of stress distribution in critical cross-section.

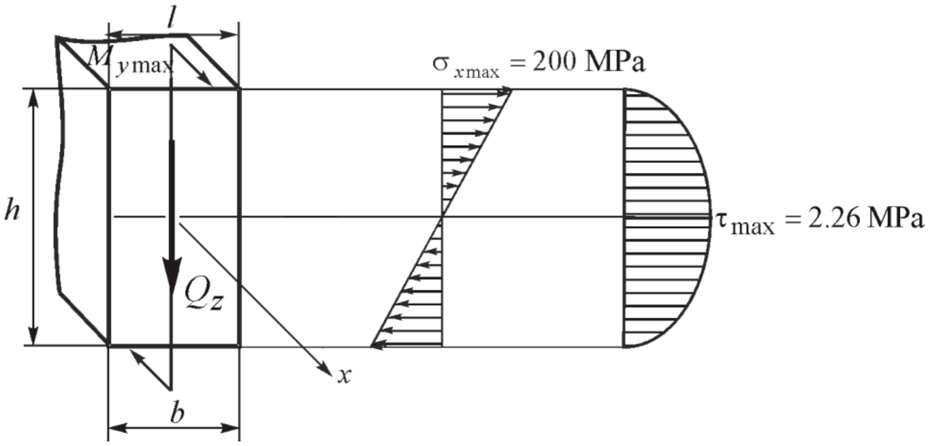


Fig. 42

$$\sigma_{\max} = \frac{\left|M_y\right|_{\max}}{W_y} = \frac{6 \times 18.2 \times 10^3}{5.15 \times 10^{-2} \times \left(10.3 \times 10^{-2}\right)^2} = 200\,\text{MPa};$$

$$\tau_{\max} = \frac{Q_z S_y^{1/2}}{bI_y} = \frac{3}{2}\frac{Q_z}{A} = \frac{3}{2} \times \frac{8 \times 10^3}{53.05 \times 10^{-4}} = 2.26\,\text{MPa}.$$

5) Calculating the diameter of round section using condition of strength.

$$W_y = \frac{\pi d^3}{32} \geq 91 \times 10^{-6}\,\text{m}^3.$$

$$d \geq \sqrt[3]{\frac{32W_y}{\pi}} = \sqrt[3]{\frac{32 \times 91 \times 10^{-6}}{3.14}} = 9.75 \times 10^{-2}\,\text{m}.$$

$$A = \frac{\pi d^3}{4} = \frac{3.14 \times 9.75^2}{4} \cdot 10^{-4} = 74.6 \times 10^{-4}\,\text{m}^2.$$

Draw the graphs of stress distributions in critical cross-sections.

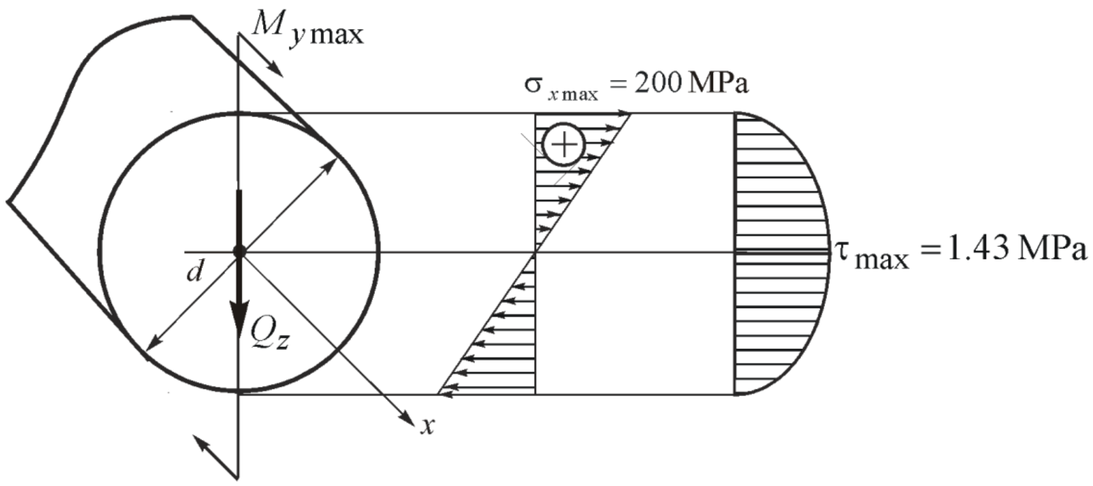


Fig. 43

$$\sigma_{\max} = \frac{\left|M_y\right|_{\max}}{W_y} = \frac{32\times18.2\times10^3}{3.14\times\left(9.75\times10^{-2}\right)^3} = 200\,\text{MPa};$$

$$\tau_{\max} = \frac{Q_z S_y^{1/2}}{bI_y} = \frac{4}{3}\frac{Q_{zx}}{A} = \frac{4}{3}\times\frac{8\times10^3}{74.6\times10^{-4}} = 1.43\,\text{MPa}.$$

6) Comparing the cross-section areas:

$$\overset{\text{I}}{A} < \overset{\blacksquare}{A} < \overset{\oslash}{A}: \quad 20.2\times10^{-4} < 53.05\times10^{-4} < 74.6\times10^{-4}\,\text{m}^2.$$

Note, that the I-beam section has the largest strength- to-weight ratio.

MINISTRY OF EDUCATION AND SCIENCE OF UKRAINE

National aerospace university “Kharkiv Aviation Institute”

Department of aircraft strength

Course
Mechanics of materials and structures

HOME PROBLEM 9

Stress Analysis of Two Supported Beams in Plane Bending

Name of student:

Group:

Advisor:

Data of submission:

Mark:

National aerospace university
"Kharkiv Aviation Institute"
Department of aircraft strength

Subject: mechanics of materials
Document: home problem
Topic: Stress Analysis of Two Supported Beams in plane Bending.

Full name of the student, group

Variant: 1 **Complexity: 1**

Given: $[\sigma]_t$ =160 MPa; $[\sigma]_c = 200$ MPa; $h/b = 2$ for rectangle cross-section.

Goal:
1) copy from home problem No5 the graphs of shear forces and bending moments ;

2) using condition of strength in pure bending calculate: a) diameter of round solid cross-section; b) diameters of hollow tube cross-section using thickness ratio $\alpha = d/D = 0{,}8$; c) dimensions of rectangle solid cross-section in $h/b = 2$; d) dimensions of hollow rectangle cross-section in $H/h = 2$; $B/b = 2$; e) number of I-beam section;

3) compare the weights of 5 cross-sections mentioned in p. 2;

4) design the graphs of acting stresses in cross-section with the largest shear force for 5 cross-sections mentioned in p.2;

5) estimate the type of stress state in the following points of I-beam section: a) lying on neutral axis; b) belonging to the most tensile or compressed layers of the section (choose yourself); c) in the point of the flange and web connection (one of two existing connections). Note, that the point must belong to the web.

Full name of the lecturer **signature**

Mark:

Given: $q = 10$ kN/m, $M = 20$ kNm, $P = 30$ kN, $a = 2$ m, $b = 4$ m, $c = 2$ m, $[\sigma]_t = 160$ MПa, $[\sigma]_c = 200$ MPa, $h/b = 2$.

Note 1. Stress analysis is possible only after calculation of internal forces in cross-sections of the beam.

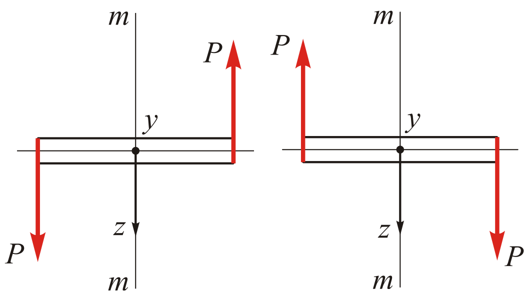
Note 2. In shear force calculating, we will use the rule that internal shear force in particular cross-section numerically equals to algebraic sum of external forces projections on z axis, but only for the forces applied to the left or to the right part of the beam.

In bending moment calculating, we will use the rule that internal bending moment in particular cross-section numerically equals to algebraic sum of external force moments, but only for the forces and moments applied to the left or to the right part of the beam.

Solution

1. Sign conventions:

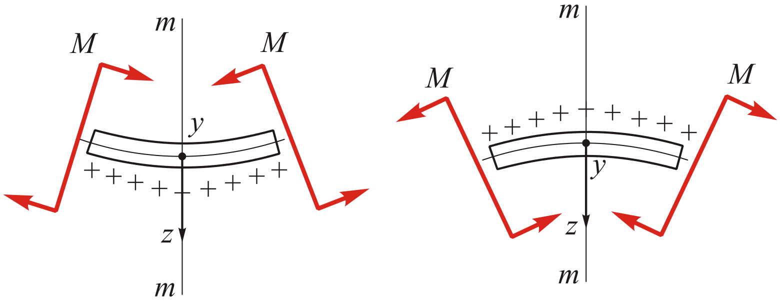
a) for shear forces (Fig. 1)



$Q_z^{m-m} < 0$ $Q_z^{m-m} > 0$

Fig. 1

b) for bending moments (Fig. 2)



$M_y^{m-m} > 0$ $M_y^{m-m} < 0$

Fig. 2

2. Calculating the reactions in supports R_A and R_C (see Fig. 3). Let us direct preliminary these reactions upwards, since their actual direction are unknown. Plus sign in solution will mean that really these reactions are directed upwards. Secondly, to determine R_A and R_C, we will use both equations of momentum balance (for example, relative to C and A points). Third equation of the force equilibrium will be used to check the result accuracy. In writing the equations of the moment balance, clockwise rotation will be assumed to be positive.

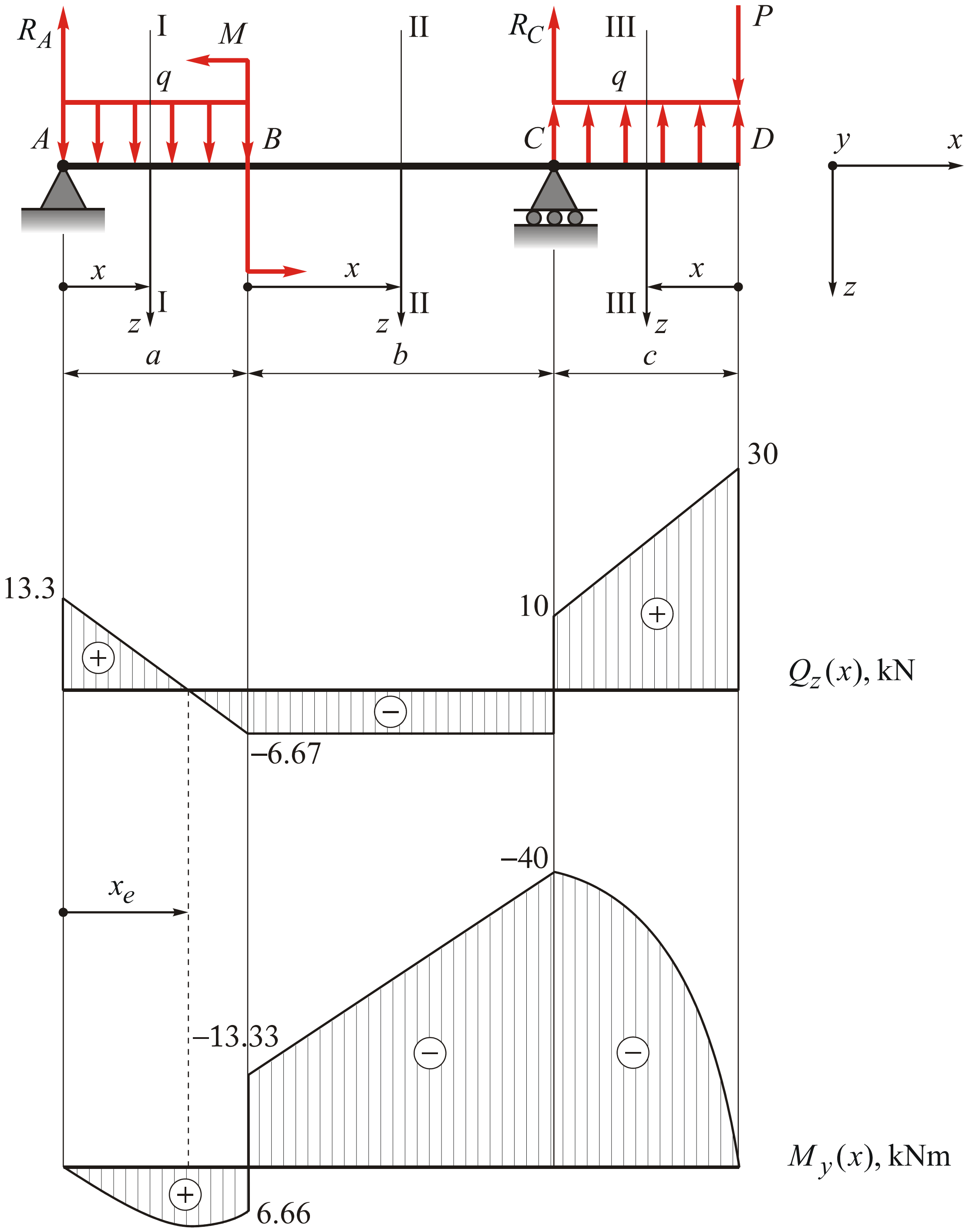


Fig. 3

$$\sum M_A = 0 = +\frac{qa^2}{2} - M - R_C(a+b) - qa(\frac{a}{2}+b+c) + P(a+b+c),$$

$$R_C = \frac{1}{a+b}\left(-\frac{qa^2}{2} + M + qa(\frac{a}{2}+b+c) - P(a+b+c)\right) = +16{,}67\text{ kN}.$$

$$\sum M_C = 0 = -\frac{qc^2}{2} - M + R_A(a+b) - qa(\frac{a}{2}+b) + Pc,$$

$$R_A = \frac{1}{a+b}\left(+\frac{qc^2}{2} + M + qa(\frac{a}{2}+b) - Pc\right) = +13{,}33\text{ kN}.$$

$$\sum P_z = 0 = -R_A - R_C - qc + qa + P = -13{,}33 - 16{,}67 - 10\times 2 + 10\times 2 + 30 = 0.$$

3. Determining the shear forces and bending moments in an arbitrary cross-sections of the beam. Two potions will be considered from the left and the last one from the right to get the simplest shape of equations.

$$\text{I} - \text{I} \quad 0 < x < a:$$

$$Q_z^I(x) = R_A - qx\,|_{x=0} = 13{,}33\,|_{x=2} = 13{,}33 - 20 = -6{,}67 \text{ kN},$$

$$M_y^I(x) = R_A x - \frac{qx^2}{2}\,|_{x=0} = 0\,|_{x=2} = 26{,}66 - 20 = 6{,}66 \text{ kNm}.$$

Note, that the change of shear force sign within this potion boundaries really means the presence of extreme value of internal bending moment within the boundaries of the potion. First of all, let us determine the coordinate of the cross-section with extreme bending moment. For this purpose, let us equate to zero the shear force equation:

$$Q_z^I(x_e) = 0 = R_A - qx_e = 13.33 - 10x_e, \quad x_e = 1.33 \text{ m (see Fig. 3)}.$$

Substituting this coordinate into bending moment equation leads to the following value:

$$M_{y_{\max}}^I = M_y^I(x_e) = R_A x_e - \frac{qx_e^2}{2} = 13.33 \times 1.33 - \frac{10}{2} \times 1.33^2 = +8{,}89 \text{ kNm}.$$

$$\text{II} - \text{II} \quad 0 < x < b:$$

$$Q_z^{II}(x) = R_A - qa = 13.33 - 20 = -6.67 \text{ kN},$$

$$M_y^{II}(x) = R_A(a + x) - qa\left(\frac{a}{2} + x\right) - M\,|_{x=0} = 26.66 - 20 - 20 =$$

$$= 13.34\,|_{x=4} = 79.98 - 100 - 20 = -40 \text{ kNm}.$$

$$\text{III} - \text{III} \quad 0 < x < c:$$

$$Q_z^{III}(x) = P - qx\,|_{x=0} = 30\,|_{x=2} = 30 - 20 = 10 \text{ kN},$$

$$M_y^{III}(x) = -Px + \frac{qx^2}{2}\,|_{x=0} = 0\,|_{x=2} = -60 + 20 = -40 \text{ kNm}.$$

4. Designing the shear force and bending moment graphs. For shear force graph positive values will be drawn upwards and vice versa. The bending moment graph will be drawn on tensile fibers (see Fig. 2).

In design problem solution, we will omit the shear forces due to their negligible influence on prismatic beam strength. In such case, we will determine critical section as the section with maximum magnitude of bending moment. In our problem, this cross-section is situated on the right support:

$$\left|M_{y_{\max}}\right| = 40 \text{ kNm}.$$

5. Calculating the sectional modulus $W_{n.a.}$ from condition of strength in critical cross-section:

$$\sigma_{\max} = \frac{|M_{y\max}|}{W_{n.a.}} \le [\sigma].$$

Note. In the case when allowable stresses are different in tension and compression, lesser value of allowable stress should be used in condition of strength, i.e. $[\sigma]_t = 160\text{ MPa}$. Then

$$W_{n.a.} = \frac{|M_{y\max}|}{[\sigma]_t} = \frac{40 \cdot 10^3}{160 \cdot 10^6} = 250 \times 10^{-6}\text{ m}^3.$$

6. Selecting the I-beam section number from the assortment.

(a) let us chose, at first, lesser number No.22 with $W_{n.a.} = 232 \times 10^{-6}\text{ m}^3$:

$$\text{No.22} \to \sigma_{\max} = \frac{40 \times 10^3}{232 \times 10^{-6}} = 172.4\text{ MPa}.$$

This number will be evidently overstressed but five percent overstress is available in mechanics of materials. It's calculating shows, that

$$\Delta = \frac{\sigma_{\max} - [\sigma]}{[\sigma]} \times 100\% = \frac{172.4 - 160}{160} \times 100\% = 7.5\%.$$

Since overstress is more than 5%, No.22 is not applicable. Therefore, larger number should be selected: No.22^a with $W_{n.a.} = 254 \times 10^{-6}\text{ m}^3$. Maximum normal stress in this I-beam section is $\sigma_{\max} = \dfrac{40 \times 10^3}{254 \times 10^{-6}} = 157.48\text{ MPa}$.

For further calculation, copy from the assortment the following dimensions and geometrical properties of No.22^a section:

$h = 22 \times 10^{-2}$ m, $b = 12 \times 10^{-2}$ m, $t = 0.89 \times 10^{-2}$ m, $d = 0.54 \times 10^{-2}$ m,

$I_y = 2790 \times 10^{-8}\text{ m}^4$, $W_y = 254 \times 10^{-6}\text{ m}^3$, $S_y^* = 143 \times 10^{-6}\text{ m}^3$, $A^I = 32.8 \times 10^{-4}\text{ m}^2$.

Note that y-axis is horizontal central axis for the section, which is really neutral axis in vertical bending. This section is shown on Fig. 4.

(b) design the graph of stress distribution in critical section under $|M_{y\max}| = 40\text{ kNm}$ and $|Q_z| = 10\text{ kN}$ loading:

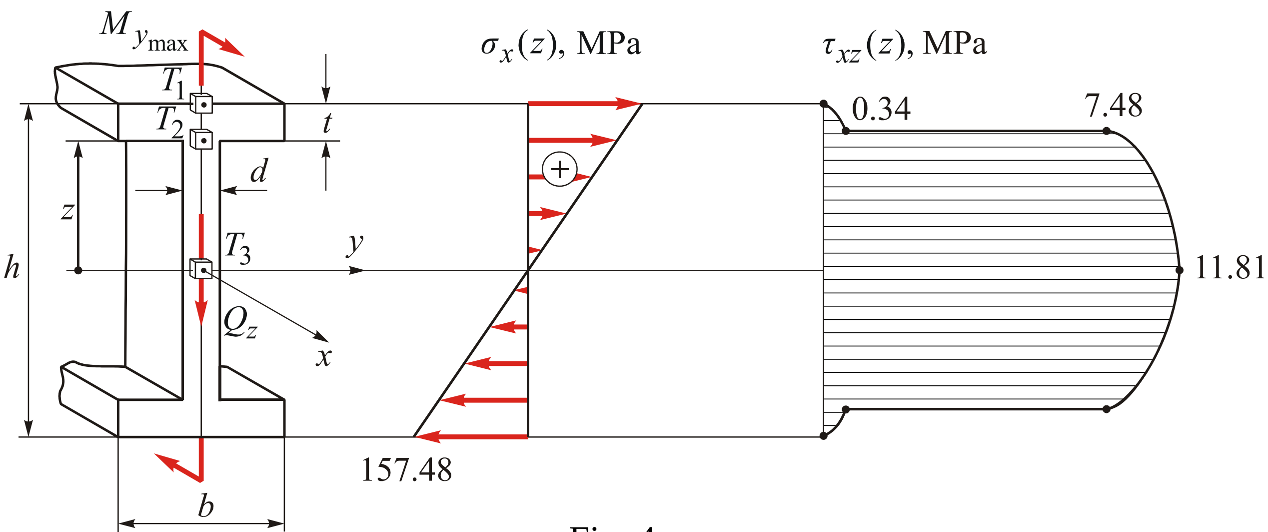


Fig. 4

To determine shear stresses and draw correspondent graph of their distribution, we will use the Juravsky formula. Knowing the stresses in three points: T_1 (outer point of the flange), T_2 (flange-web connection), T_3 (point of neutral axis), it becomes possible to draw parabolic graph of stress distribution.

$$\tau_1 = 0.$$

$$\tau_{2(flange)} = \frac{Q_z bt\left(\frac{h}{2}-\frac{t}{2}\right)}{bI_y} = \frac{10\times10^3\times12\times10^{-2}\times0.89\times10^{-2}\left(\frac{22\times10^{-2}}{2}-\frac{0.89\times10^{-2}}{2}\right)}{12\times10^{-2}\times2790\times10^{-8}} =$$

$$= 0.34 \text{ MPa.}$$

Note, that the flange width b was introduced into the Juravsky formula as the width of corresponding layer of the section.

$$\tau_{2(web)} = \frac{Q_z bt\left(\frac{h}{2}-\frac{t}{2}\right)}{dI_y} = \frac{10\times10^3\times12\times10^{-2}\cdot0.89\times10^{-2}\left(\frac{22\times10^{-2}}{2}-\frac{0.89\times10^{-2}}{2}\right)}{0.54\times10^{-2}\times2790\times10^{-8}} =$$

$$= 7.48 \text{ MPa.}$$

Note, that the web width d was introduced into the Juravsky formula as the width of corresponding layer of the section since this point belongs to the web.

$$\tau_3 = \tau_{\max} = \frac{Q_z S_y^*}{dI_y} = \frac{10\times10^3\times178\times10^{-6}}{0.54\times10^{-2}\times2790\times10^{-8}} = 11.81 \text{ MPa.}$$

Note, that the S_y^* value is the first moment of half-section relative to neutral axis of the section. It was preliminary found from assortment.

(c) analysis of the stress state type in T_1, $T_{2(web)}$, T_3 points of critical section (see Fig. 5, 6, 7):

Point T_1.

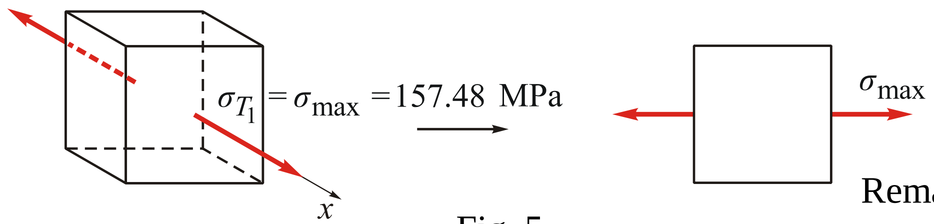


Fig. 5

Remaining principal stresses are: $\sigma_2 = 0$, $\sigma_3 = 0$

Conclusion 1: deformation is tension.
Conclusion 2: stress state is uniaxial.

Point $T_{2(web)}$.

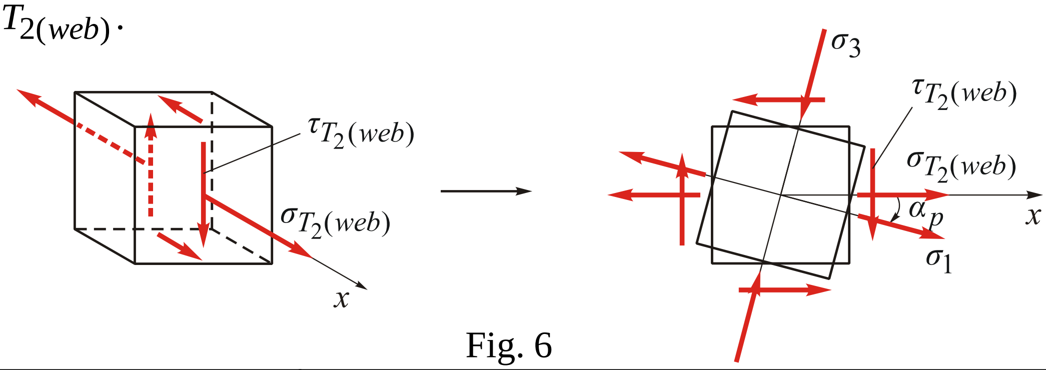


Fig. 6

$$\sigma_{T_2(web)} = \frac{M_{y_{\max}}\left(\frac{h}{2} - t\right)}{I_y} = \frac{40\times10^3\left(\frac{22\times10^{-2}}{2} - 0.89\times10^{-2}\right)}{2790\times10^{-8}} = 144.95 \text{ MPa},$$

$$\tau_{T_2(web)} = 7.48 \text{ MPa}.$$

To determine the stress-state type, let us determine principal stresses and the angle of principal planes inclination. The following formulae will be used for this purpose:

$$\sigma_{\substack{\max\\ \min}} = \frac{\sigma_\alpha + \sigma_\beta}{2} \pm \frac{1}{2}\sqrt{\left(\sigma_\alpha - \sigma_\beta\right)^2 + 4\tau_\alpha^2},$$

$$tg2\alpha_p = \frac{2\tau_\alpha}{\sigma_\beta - \sigma_\alpha}.$$

Since the condition $\sigma_\alpha > \sigma_\beta$ was assumed in these formulae proof, let us re-designate the stresses:

$$\sigma_\alpha = \sigma_{T_2(web)} = +144.95 \text{ MPa},$$

$$\sigma_\beta = 0,$$

$$\tau_\alpha = \tau_{T_2(web)} = +7.48 \text{ MPa},$$

$$\tau_\beta = -\tau_\alpha = -7.48 \text{ MPa}.$$

Then

$$\sigma_{\substack{\max\\ \min}} = \frac{\sigma_\alpha + \sigma_\beta}{2} \pm \frac{1}{2}\sqrt{\left(\sigma_\alpha - \sigma_\beta\right)^2 + 4\tau_\alpha^2} = \frac{144.95 + 0}{2} \pm \frac{1}{2}\sqrt{(144.95 - 0)^2 + 4\times7.48^2}.$$

$$\sigma_{\max} = +145.34 \text{ MPa} = \sigma_1,$$

$$\sigma_{\min} = -0.39 \text{ MPa} = \sigma_3,$$

Checking the invariability of normal stresses sum in rotation of axes:

$$\sigma_\alpha + \sigma_\beta = \sigma_1 + \sigma_3 \;\rightarrow\; +144.95 + 0 = +145.34 - 0.39.$$

Calculation of the principal planes inclination:

$$\text{tg}2\alpha_p = \frac{2\tau_\alpha}{\sigma_\beta - \sigma_\alpha} = \frac{2(+7.48)}{0 - 144.95} = -0.1032,$$

$$2\alpha_p = -5.9° \;\rightarrow\; \alpha_0 = -2.95° \text{ (clockwise rotation).}$$

Conclusion: stress state is plane (biaxial) (see Fig. 6).

Point T_3.

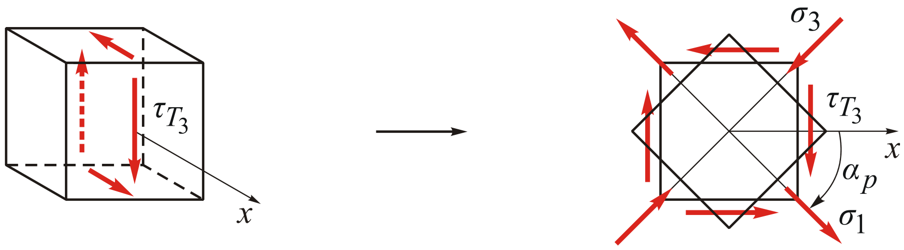


Fig. 7

In this case, $\sigma_{T_3} = 0$ MPa, $\tau_{T_3} = \tau_{\max} = 11.81$ MPa.

Conclusion: deformation type is pure shear, stress state is plane (biaxial) (see Fig. 7).

Calculation of principal stresses and the angle of principal planes inclination.

$$\sigma_{\substack{max\\min}} = \frac{\sigma_\alpha + \sigma_\beta}{2} \pm \frac{1}{2}\sqrt{\left(\sigma_\alpha + \sigma_\beta\right)^2 + 4\tau_\alpha^2},$$

Since the condition $\sigma_\alpha > \sigma_\beta$ was assumed in these formulae proof, let us re-designate the stresses:

$$\sigma_\alpha = \sigma_\beta = 0,\ \tau_\alpha = +\tau_{T_3} = +11.81\,\text{MPa},\ \tau_\beta = -\tau_\alpha = -11.81\,\text{MPa}.$$

After substituting,

$$\sigma_{max} = +\tau_\alpha = +11.81\,\text{MPa} = \sigma_1,$$
$$\sigma_{min} = -\tau_\alpha = -11.81\,\text{MPa} = \sigma_3,$$
$$\sigma_2 = 0.$$

Calculation of the principal planes inclination:

$$\operatorname{tg} 2\alpha_p = \frac{2\tau_\alpha}{\sigma_\beta - \sigma_\alpha} = \frac{2(+11.81)}{0-0} = -\infty,\ 2\alpha_p = -90°,\ \alpha_p = -45°\ \text{(clockwise rotation)}$$

Principal stresses are shown on Fig. 7.

Conclusion: stress state is biaxial, deformation is pure shear.

7. Calculating the round cross-section diameter.

(a) it was found earlier from the condition of strength that $W_y = 250 \times 10^{-6}\ \text{m}^3$. On the other hand,

$$W_y^{\otimes} = \frac{\pi D^3}{32} \rightarrow D = \sqrt[3]{\frac{32W_y}{\pi}} = \sqrt[3]{\frac{32 \times 250 \times 10^{-6}}{3.14}} = 13.66 \times 10^{-2}\ \text{m}.$$

(b) cross-sectional area is

$$A^{\otimes} = \frac{\pi D^2}{4} = \frac{3.14(13.66 \times 10^{-2})^2}{4} = 146.48 \times 10^{-4}\ \text{m}^2.$$

Note, that this area is significantly more than the area of corresponding I-beam section: $A = 32.8 \times 10^{-4}\ \text{m}^2$.

(c) draw the graphs of stress distribution in critical section:

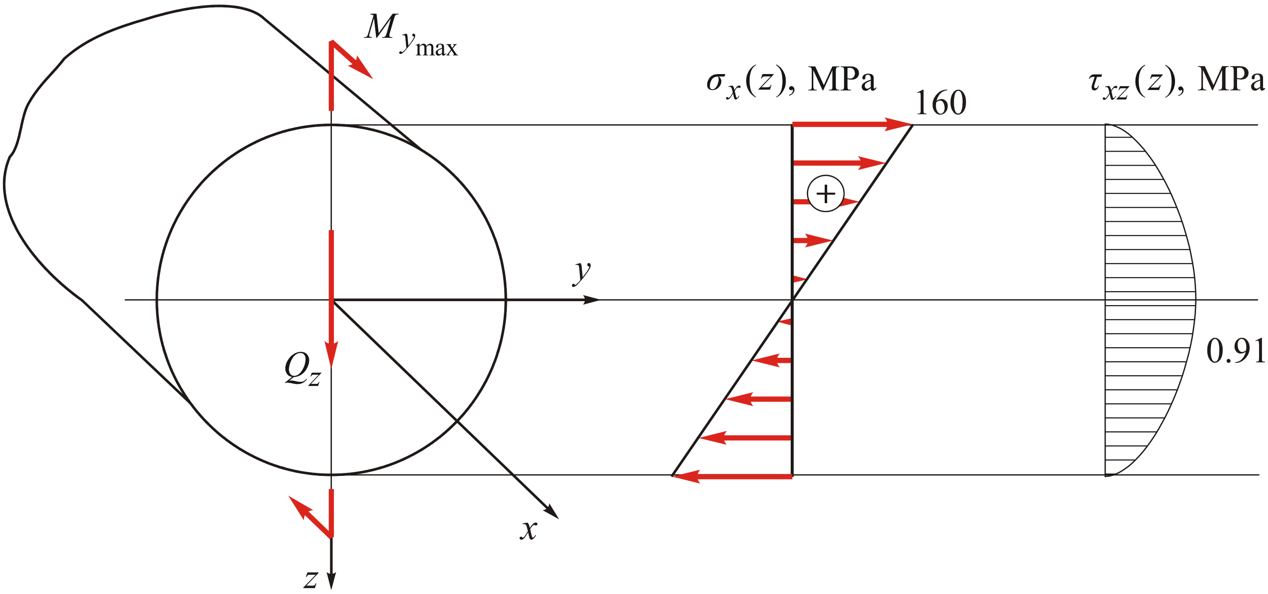


Fig. 8

$$\sigma_{max} = \frac{32M_{y\,max}}{\pi D^3} = \frac{32 \times 40 \times 10^3}{3.14 \times (13.66 \times 10^{-2})^3} = 160\ \text{MPa},$$

$$\tau_{max} = \frac{4}{3}\frac{Q_z}{A} = \frac{4\times 10\times 10^3}{3\times 146.48\times 10^{-4}} = 0.91 \text{ MPa}.$$

8. Calculating the dimensions for rectangle cross-section.
Let us assume, that $h/b = 2$.
(a) from condition of strength the sectional modulus should be equal to $W_y = 250\times 10^{-6}$ m^3. From the other hand, $W_y = \frac{bh^2}{6}$. After substituting $h = 2b$, we get

$$\frac{4b^3}{6} = 250\times 10^{-6} \text{ m}^3,$$

$$b \geq \sqrt[3]{\frac{3W_y}{2}} = \sqrt[3]{\frac{3\times 250\times 10^{-6}}{2}} = 7.22\times 10^{-2} \text{ m},$$

$$h = 14.44\times 10^{-2} \text{ m}.$$

(b) calculation of cross-section area:

$$A = bh = 104.26\times 10^{-4} \text{ m}^2.$$

Note, that this area is less than the area of round section but more than the area of I-beam section.

(c) draw the graphs of stress distribution in critical section:

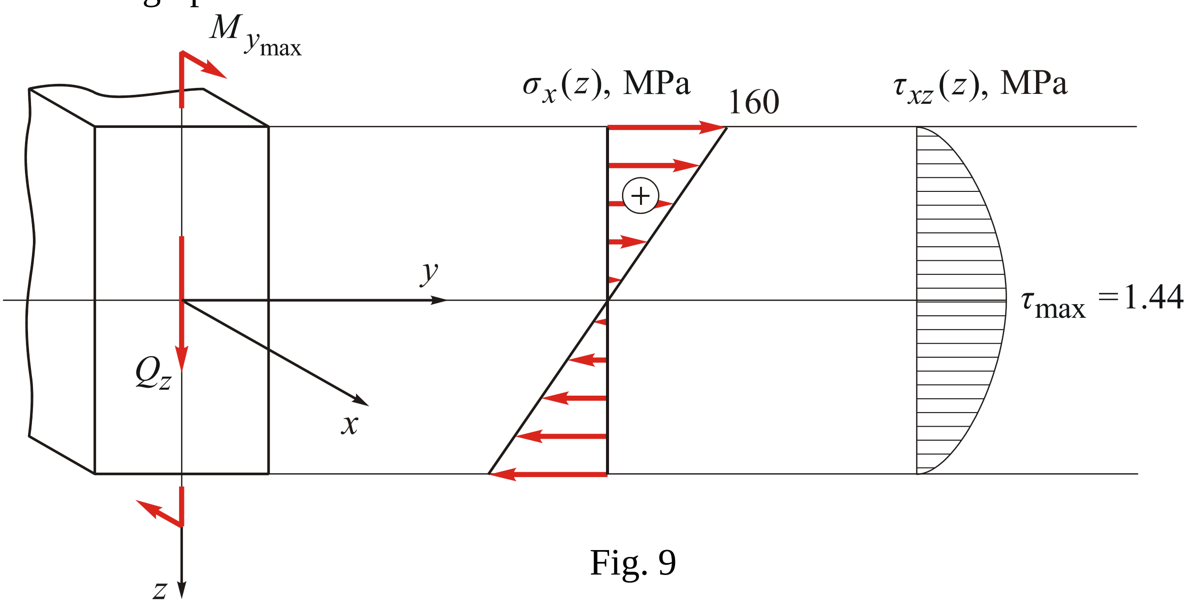


Fig. 9

$$\sigma_{max} = \frac{6M_{y_{max}}}{bh^2} = \frac{6\times 40\times 10^3}{7.22\times 10^{-2}\times\left(14.44\times 10^{-2}\right)^2} = 160 \text{ MPa},$$

$$\tau_{max} = \frac{3}{2}\frac{Q_z}{A} = \frac{3\times 10\times 10^3}{2\times 104.26\times 10^{-4}} = 1.44 \text{ MPa}.$$

9. General conclusions:

a) $D^{\otimes} < h^{\square} < h^{I}$ $(13.66\times 10^{-2} \text{ m} < 14.44\times 10^{-2} \text{ m} < 22\times 10^{-2} \text{ m})$;

b) $\sigma_{max}^{\otimes} = \sigma_{max}^{\square} \cong \sigma_{max}^{I}$ $(160 \text{ MPa} = 160 \text{ MPa} \cong 157.48 \text{ MPa})$;

c) $\tau_{max}^{\otimes} < \tau_{max}^{\square} \ll \tau_{max}^{I}$ $(0.91 \text{ MPa} < 1.44 \text{ MPa} \ll 11.81 \text{ MPa})$;

d) $A^{\otimes} > A^{\square} > A^{I}$ $(146.48\times 10^{-4} \text{ m}^2 > 104.26\times 10^{-4} \text{ m}^2 > 32.8\times 10^{-4} \text{ m}^2)$.