LECTURE 11 Strength and Rigidity of a Bar in Tension and Compression

Tension and compression are the types of simple deformation in which only one internal force factor, a normal force N_x, appears in the cross section of a bar: $N_x \neq 0$, $Q_z = Q_y = M_x = M_y = M_z = 0$. The examples of tension-compression deformation of structural elements are represented in Figs 1–16.

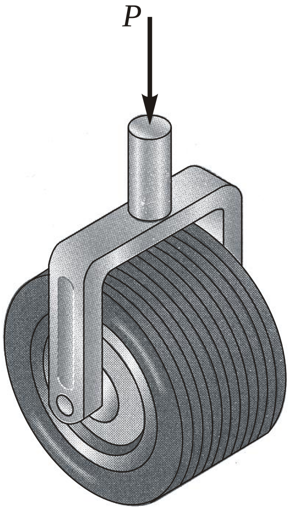


Fig. 1 The vertical load *P* acting on the wheel compresses its vertical axis

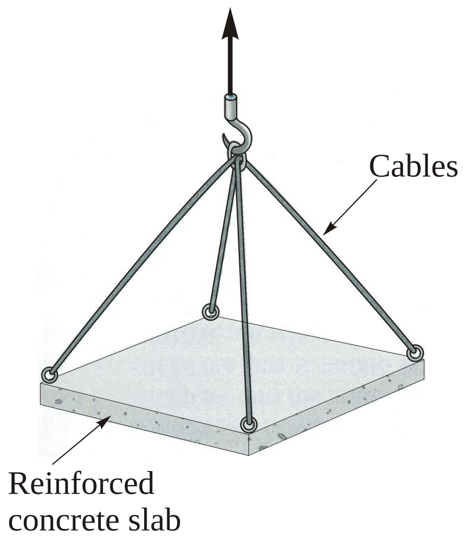


Fig. 2 A reinforced concrete slab is lifted by 4 tensile cables

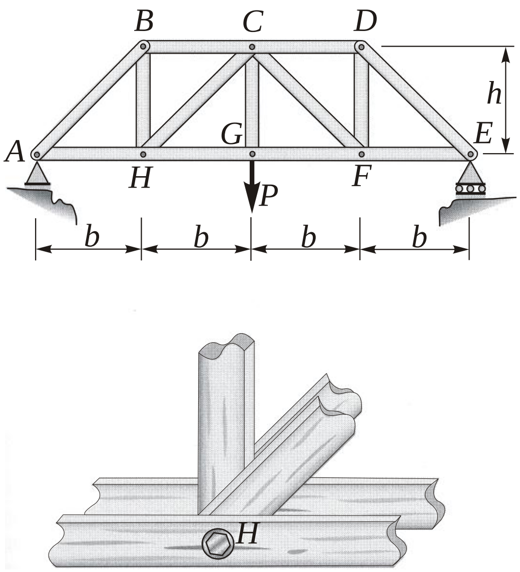


Fig. 3 The truss *ABCDEFGH* is part of a wood bridge. The truss members are hinged

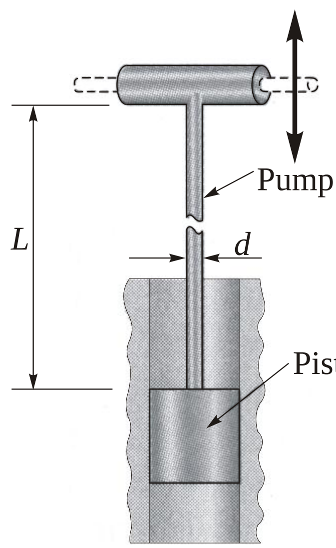


Fig. 4 A pump moves a piston up and down in a deep water well. Pump rod is periodically tensile or compressed

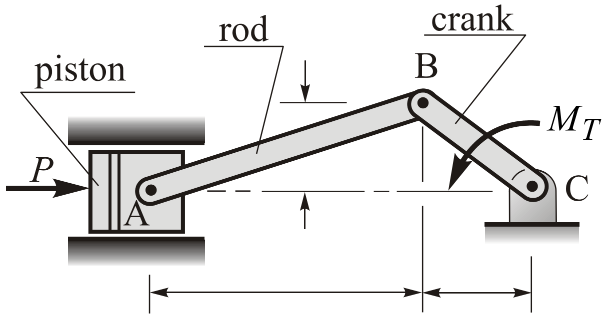


Fig. 5 The piston in an engine is attached to a connecting rod *AB*, which intern is connected to a crank arm *BC*. Both elements are compressed or tensile

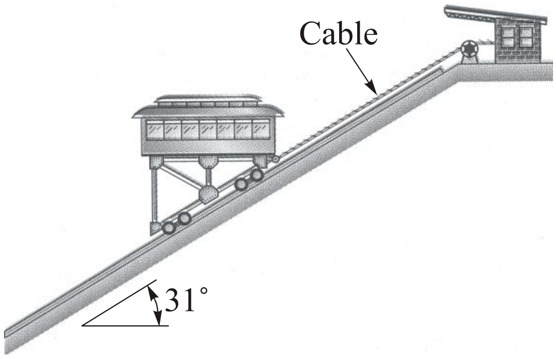


Fig. 6 A car is pulled slowly up a steep inclined track by a steel cable which is tensile

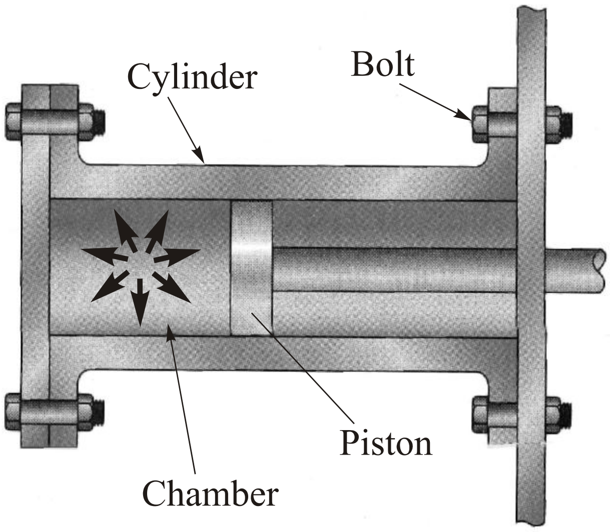


Fig. 7 Cylinder with piston and clamping tensile bolts

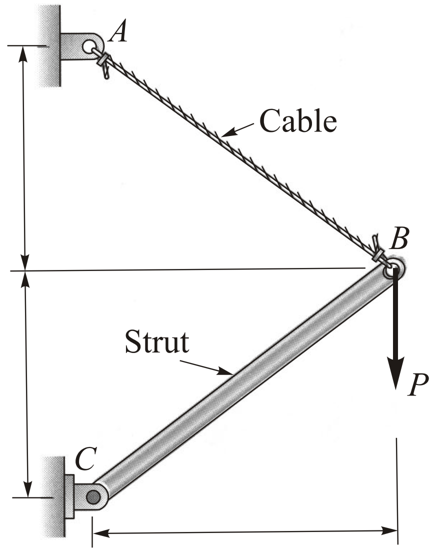


Fig. 8 A cable and strut assembly *ABC* supports a vertical load *P.* The cable and strut are in tension-compression.

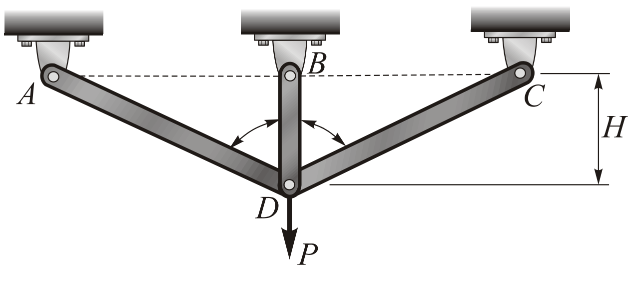


Fig. 9 A vertical load *P* is supported by the truss *ABCD* which are pinned connected and

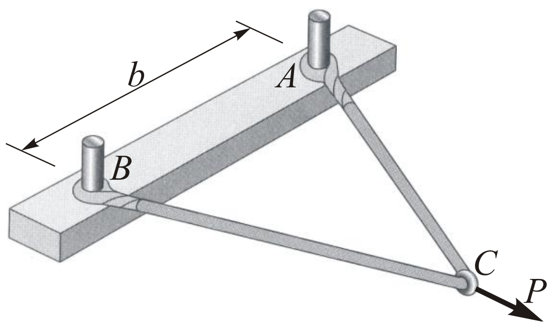


Fig. 10 A bungee cord is attached to pegs

therefore are tensile or compressed

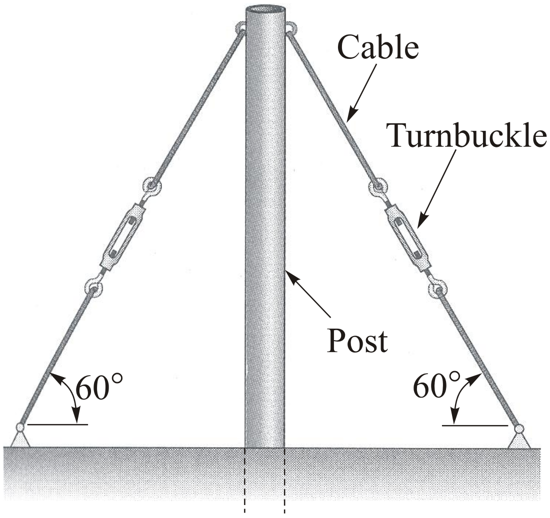


Fig. 11 A tubular post is guyed by two cables fitted with turnbuckles. The cables are tightened by rotating the turnbuckles

and pulled at its midpoint by a force *P*

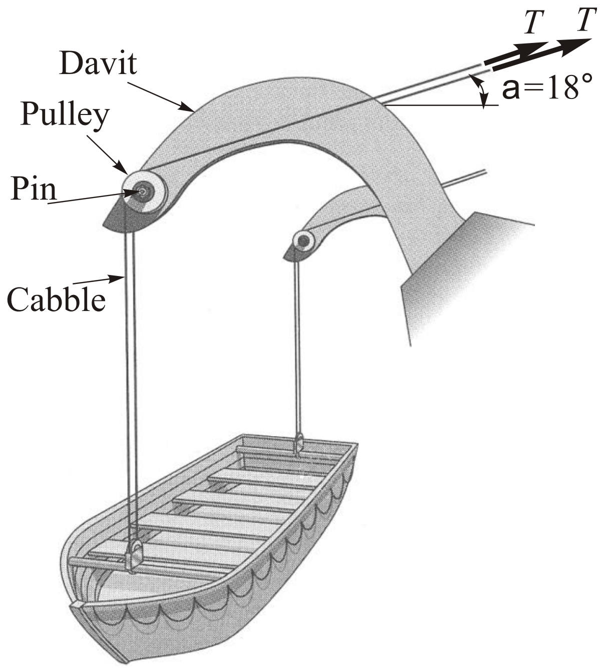


Fig. 12 Lifeboat hanges from two ship's davits. Cables attached to the lifeboat pass over the pulleys and are really tensile

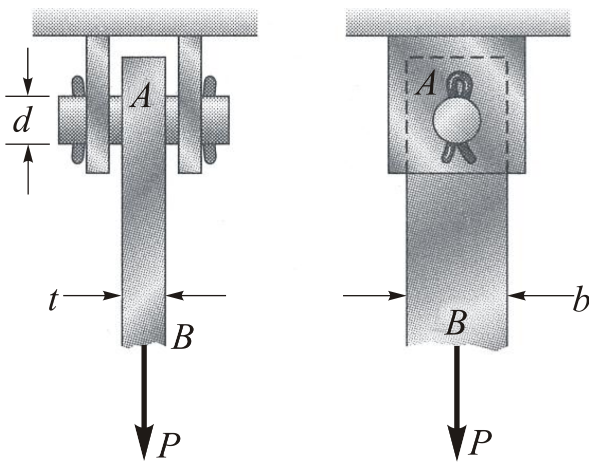


Fig. 13 An aluminum bar *AB* is attached to a support by a pin and is tensile by a *P* force

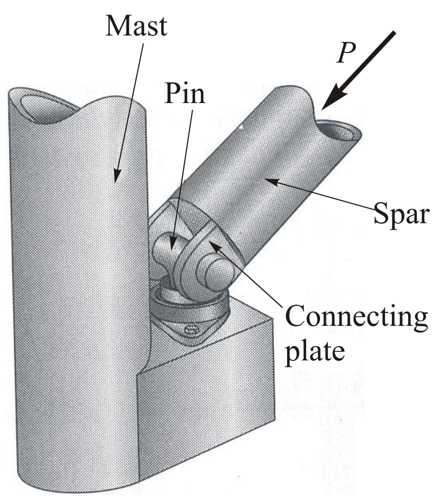


Fig. 14 A ship's spar is attached at the base of a must by a pin connection. It is compressed by a *P* force

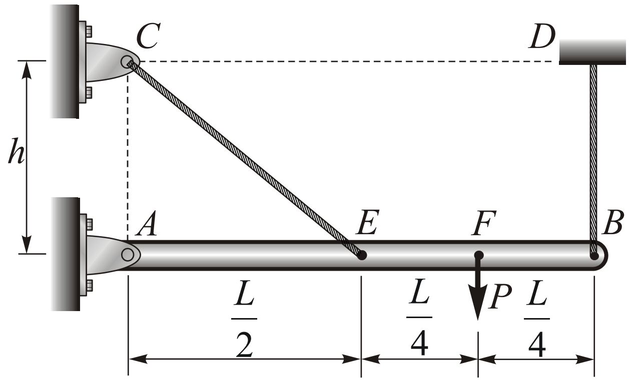


Fig. 15 A rigid bar *AB* is supported by two cables *CE* and *BD* which are tensile under *P* loading

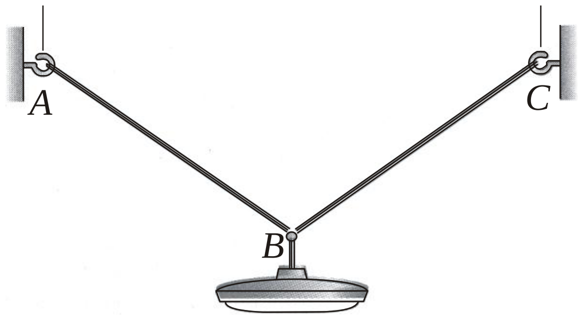


Fig. 16 A steel wire *ABC* supporting a lamp is tensile

1 Hypothesis of Plane Sections

If before loading the ends of the segments (left points *a*, *c*, *e* and also right ones *b*, *d*, *f*) (see Fig. 15) are situated at the planes *A* and *B*, then after loading they also will be situated at the planes, accordingly, A' and B' (left points a', c', e' and also right ones b', d', f'). This fact is named as **hypothesis of plane sections**, i.e. $A' \| A$ and $B' \| B$. It is accepted as fundamental and confirmed by experimental tests.

Note the experimentally proven fact that absolute elongations for all elementary segments (*ab*, *cd*, *ef*,...) taken in the portion *dx* are the same

$$\Delta_{ab} = \Delta_{cd} = \Delta_{ef} = \Delta dx, \qquad \text{and strains} \qquad \varepsilon_x = \frac{\Delta dx}{dx} = const, \tag{1}$$

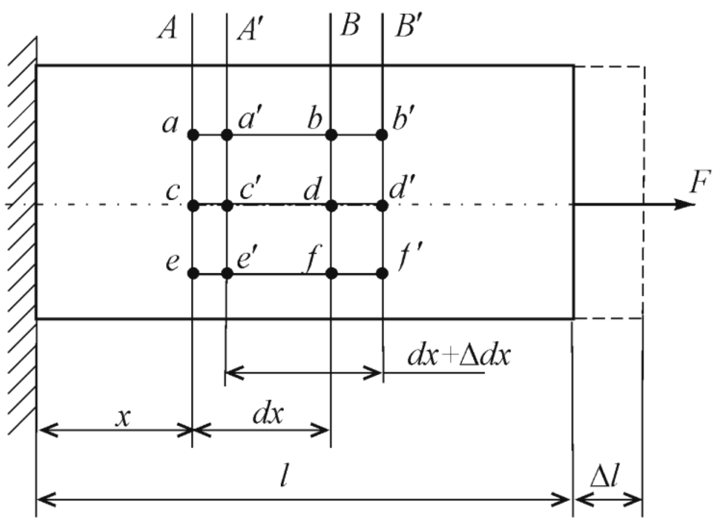


Fig.17

i.e. the ***state of strain is homogeneous over the cross-section of loaded bar***.

2 Calculation of Stresses in Tension-Compression

It was mentioned above, that relation between normal force N_x and normal stress σ_x is

$$N_x = \int_A \sigma_x dA. \tag{2}$$

According to Hooke's law

$$\sigma_x = E\varepsilon_x, \tag{3}$$

($E = const$) and taking into account the expression (1), we have

$$N_x = \int_A E\varepsilon_x dA = E\varepsilon_x A = \sigma_x A.$$

Then

$$\sigma_x = \frac{N_x}{A}. \tag{4}$$

As it is seen from expression (4) the ***normal stresses in tension (or compression) are distributed uniformly over the cross-section of a rod***. The formula (4) means, that to find acting stresses in the cross-section it is necessary to know from normal force diagram the value of corresponding normal force in the cross-section and also cross-sectional area of the rod.

3 Condition of Strength in Tension-Compression

To estimate the strength of the bar in tension and compression, it is necessary to create the **condition of strength** which states that the ***maximum stress*** $\boldsymbol{\sigma_{\max}}$ ***in the bar must not exceed the allowable stress*** of the bar material $[\sigma]$:

$$\sigma_{\max t} \le [\sigma]_t, \quad \text{and} \quad \sigma_{\max_c} \le [\sigma]_c, \quad \text{where} \tag{5}$$

$[\sigma]_t$ – allowable stress in tension; $[\sigma]_c$ – allowable stress in compression.

The section of the bar in which the stress $\sigma_{\max}$ acts is called the **critical section**. Noting that

$$\sigma_{\max} = \left(\frac{N_x}{A}\right)_{\max} \tag{6}$$

and following the condition of strength

$$\sigma_{\max} = \left(\frac{N_x}{A}\right)_{\max} \le [\sigma], \tag{7}$$

it is possible to solve three practical problems as follows:

(1) check the strength of a bar for the specified load and cross-sectional area using condition

$$\sigma_{\max} \le [\sigma] \quad \text{or} \quad \sigma_{\max} = \left(\frac{N_x}{A}\right)_{\max} \le [\sigma]; \tag{8}$$

(2) determine the cross-sectional area A for the specified load and allowable stress $[\sigma]$:

$$A \ge N_x / [\sigma]; \tag{9}$$

(3) determine the allowable load on a bar for specified cross-sectional area and allowable stress taking into account that

$$[N_x] = A[\sigma], \tag{10}$$

where $[N_x]$ – allowable normal force.

4 Determination of Strains

According to expression (3) $\sigma_x = E\varepsilon_x$.

Replace σ_x by N_x / A and ε_x by $\dfrac{\Delta dx}{dx}$ (see Fig. 17). We obtain then

$$\Delta dx = \frac{N_x dx}{EA}. \tag{11}$$

The total elongation of the bar along the length *l* is

$$\int_l \Delta dx = \int_l \frac{N_x dx}{EA} \quad \text{or} \quad \Delta l = \int_0^l \frac{N_x dx}{EA}. \tag{12}$$

When the bar is loaded at the ends of portions only, then normal force $N_x = F$ is independent of *x*. If in addition, the bar has constant cross-sectional area *A*, we obtain from expression (12):

$$\Delta l = \frac{N_x l}{EA}. \tag{13}$$

For stepped bar total elongation will be as follows:

$$\Delta l = \sum_{i=1}^{n} \Delta l_i = \sum_{i=1}^{n} \frac{N_{x_i} l_i}{E_i A_i}, \tag{14}$$

where *i* is number of portion.

Internal loads are not the only sources of stresses and strains in a structure. Changes in temperature produce expansion or contraction of the material, resulting in **thermal strains** and **thermal stresses**. A simple illustration of thermal expansion is shown in Fig. 18, where the block of material is unrestrained and therefore free to expand. When the block is heated, every element of the material undergoes thermal strains in all directions, and consequently the dimensions of the block increase. If we take corner *A* as a fixed reference point and let side *AB* maintain its original alignment, the block will have the shape shown by the dashed lines.

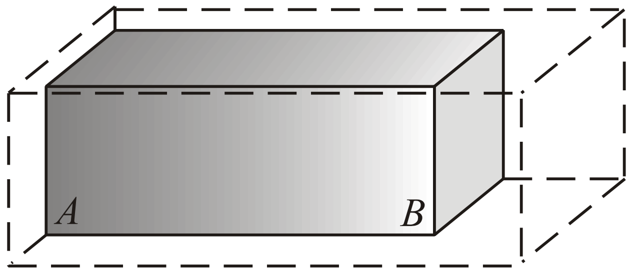


Fig. 18 Block of material subjected to an increase in temperature

For most structural materials, thermal strains is proportional to the temperature change. In solving many practical problems, ***elongations due to temperature effects should be taken into account as well***

as elongations resulting from stress $\boldsymbol{\sigma}$. In this case the method of superposition is used and total elongation may be written as the sum:

$$\Delta l\left(N,\Delta t\right)=\frac{N_x l}{EA}+\alpha_t\cdot\Delta t\cdot l\,, \tag{15}$$

where α_t is the **linear coefficient of thermal expansion** of the material; Δt is a **change of temperature**.

5 Condition of Rigidity in Tension-Compression

(a) in tension

$$\Delta l_{\max}=\left(\frac{N_x l}{EA}\right)_{\max}\le\left[\Delta l\right]; \tag{16}$$

(b) in compression

$$\left|\Delta l_{\max}\right|=\left(\frac{|N_x|\cdot l}{EA}\right)_{\max}\le\left[\Delta l\right], \tag{17}$$

where $\left[\Delta l\right]$ is the **allowable elongation**.

Example 1 Stresses and elongations in statically determinate rod in tension-compression.

Given: $F_1=10\,\text{kN}$, $F_2=40\,\text{kN}$, $F_3=60\,\text{kN}$, $a=3\,\text{m}$, $b=4\,\text{m}$, $c=5\,\text{m}$, $E=2\times10^{11}\,\text{Pa}$, $A=2\times10^{-4}\,\text{m}^2$.

R.D.: 1) calculate normal forces in the rod cross-sections and design the graph of their distribution along the rod length;

2) calculate acting stresses in the rod cross-sections and design the graph of their distribution along the rod length;

3) draw the graph of the rod cross-section displacements.

Solution

1) Calculating the normal forces in cross-sections of the rod using the method of section:

I–I $0 < x < c$

$$N_x^I(x) = 0;$$

II–II $0 < x < b/3$

$$N_x^{II}(x) = -F_1 = -10\,\text{kN};$$

III–III $0 < x < 2b/3$

$$N_x^{III}(x) = -F_1 + F_3 = -10 + 60 = +50\,\text{kN};$$

IV–IV $0 < x < a$

$$N_x^{IV}(x) = -F_1 + F_3 - F_2 = -10 + 60 - 40 = +10\,\text{kN}.$$

2) Calculating the stresses in cross-sections of the rod:

I–I $0 < x < c$

$$\sigma_x^I(x) = 0;$$

II–II $0 < x < b/3$

$$\sigma_x^{II}(x) = \frac{N_x^{II}(x)}{2A} = -\frac{10\times10^3}{2\times2\times10^{-4}} = -25\,\text{MPa};$$

III–III $0 < x < 2b/3$

$$\sigma_x^{III}(x) = \frac{N_x^{III}(x)}{2A} = +\frac{50\times10^3}{2\times2\times10^{-4}} = +125\,\text{MPa};$$

IV–IV $0 < x < a$

$$\sigma_x^{IV}(x) = \frac{N_x^{IV}(x)}{A} = +\frac{10\times10^3}{2\times10^{-4}} = +50\,\text{MPa}.$$

3) Determination of the rod elongations.

To solve this problem, we will use the following formula: $\Delta l(x) = \dfrac{N_x x}{EA} = kx$. Let us begin determining the displacements of the portions boundaries (points E, D, C, A)

calculating corresponding segments elongations. In this, we will use point B as the origin. Corresponding displacements are

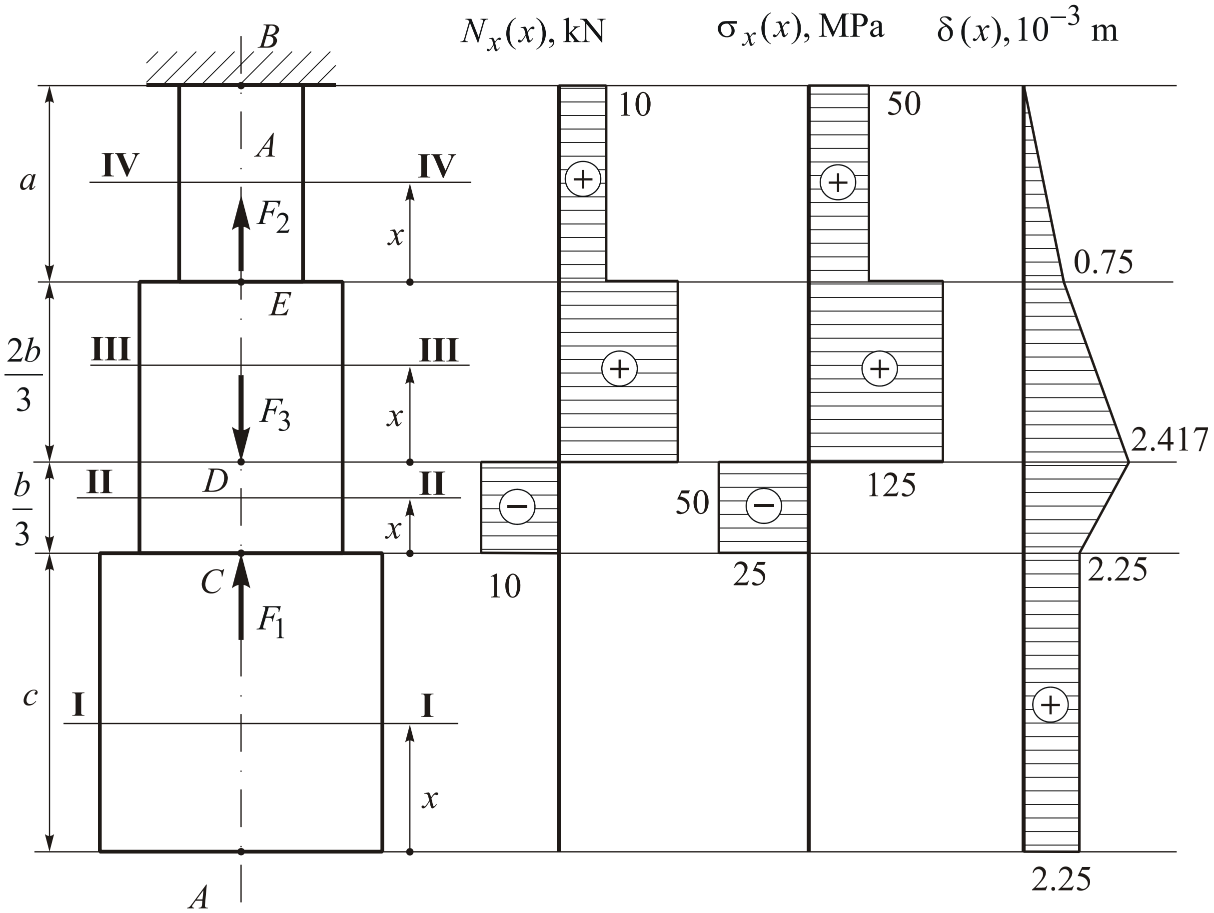


Fig. 19

$$\delta_{p.E} = \Delta l_{BE} = \frac{N_x^{IV} a}{EA} = +\frac{10 \times 10^3 \times 3}{2 \times 10^{11} \times 2 \times 10^{-4}} = +0.75 \times 10^{-3} \text{m}.$$

$$\delta_{p.D} = \Delta l_{BD} = \Delta l_{BE} + \Delta l_{ED} = +0.75 \times 10^{-3} + \frac{N_x^{III}\, 2b}{3 \times 2EA} = +0.75 \times 10^{-3} +$$

$$+\frac{50 \times 10^3 \times 2 \times 4}{3 \times 2 \times 2 \times 10^{11} \times 2 \times 10^{-4}} = +0.75 \times 10^{-3} + 1.667 \times 10^{-3} = +2.417 \times 10^{-3} \text{m}.$$

$$\delta_{p.C} = \Delta l_{BC} = \Delta l_{BD} + \Delta l_{DC} = +2.417 \times 10^{-3} + \frac{N_x^{II} b}{3 \times 2EA} = +2.417 \times 10^{-3} -$$

$$-\frac{10 \times 10^3 \times 4}{3 \times 2 \times 2 \times 10^{11} \times 2 \times 10^{-4}} = +2.417 \times 10^{-3} - 0.167 \times 10^3 = +2.25 \times 10^{-3} \text{m}.$$

$$\delta_{p.A} = \Delta l_{BA} = \Delta l_{BC} + \Delta l_{CA} = +2.25\times10^{-3} + \frac{N_x^I c}{3EA} = +2.25\times10^{-3} + 0 = +2.25\times10^{-3}\text{m}.$$

Note: "+" sign of the point displacement means the positive elongation of the corresponding segment, i.e. down directed displacement of the point. In our example, all cross-sections move downward.

Example 2 Calculation of allowable load for rod system

The contraption shown in Fig. 20a consists of a horizontal beam *ABC* supported by two vertical bars *BD* and *CE*. Bar *CE* is pinned at both ends but bar *BD* is fixed to the foundation at its lower end. The distance from *A* to *B* is 450 mm and from *B* to *C* is 225 mm. Bars *BD* and *CE* have lengths of 480 mm and 600 mm, respectively, and their cross-sectional areas are 1020 mm^2 and 520 mm^2, respectively. The bars are made of steel having a modulus of elasticity $E = 205\,\text{GPa}$. Assuming that **beam *ABC* is rigid**, find the maximum allowable load $P_{\max}$ if the displacement of point *A* is limited to 1.0 mm.

Solution To find the displacement of point *A*, we need to know the displacements of points *B* and *C*. Therefore, we must find the changes in lengths of bars *BD* and *CE*, using the general equation $\delta = NL/EA$. We begin by finding the forces in the bars from a free-body diagram of the beam (Fig. 20b). Because bar *CE* is pinned at both ends, it is a "two-force" member and transmits only a vertical force F_{CE} to the beam. However, bar *BD* can transmit both a vertical and a horizontal force. From equilibrium of beam *ABC* in the horizontal direction, we see that the horizontal force vanishes.

Two additional equations of equilibrium enable us to express the forces F_{BD} and F_{CE} in terms of the load *P*. Thus, by taking moments about point *B* and then summing forces in the vertical direction, we find

$$F_{CE} = 2P \quad \text{and} \quad F_{BD} = 3P. \tag{1}$$

Note that the force F_{CE} acts downward on bar *ABC* and the force F_{BD} acts upward. Therefore, member *CE* is in tension and member *BD* is in compression.

The shortening of member *BD* is

$$\delta_{BD} = \frac{F_{BD}L_{BD}}{EA_{BD}} = \frac{(3P)(480\ \text{mm})}{(205\ \text{GPa})(1020\ \text{mm}^2)} = 6.887P \times 10^{-6}\ \text{mm} \quad (P = \text{newtons}) \qquad (2)$$

Note that the shortening δ_{BD} is expressed in millimeters provided the load P is expressed in newtons. Similarly, the elongation of member CE is

$$\delta_{CE} = \frac{F_{CE}L_{CE}}{EA_{CE}} = \frac{(2P)(600\ \text{mm})}{(205\ \text{GPa})(520\ \text{mm}^2)} = 11.26P \times 10^{-6}\ \text{mm} \quad (P = \text{newtons}) \qquad (3)$$

Again, the displacement is expressed in millimeters provided the load P is expressed in newtons. Knowing the changes in lengths of the two bars, we can now find the displacement of point A.

A **diagram of displacements compatibility** showing the relative positions of points A, B, and C is sketched in Fig. 20c. Line ABC represents the original alignment of the three points. After the load P is applied, member BD shortens by the amount δ_{BD} and point B moves to B'. Also, member CE elongates by the amount δ_{CE} and point C moves to C'. Because the beam ABC is assumed to be rigid, points A', B', and C' lie on a straight line.

For clarity, the displacements are highly exaggerated in the diagram. In reality, line ABC rotates through a very small angle to its new position $A'B'C'$.

Using similar triangles, we can now find the relationships between the displacements at points A, B, and C. From triangles $A'A''C'$ and $B'B''C'$ we get

$$\frac{A'A''}{A''C'} = \frac{B'B''}{B''C'} \quad \text{or} \quad \frac{\delta_A + \delta_{CE}}{450 + 225} = \frac{\delta_{BD} + \delta_{CE}}{225} \qquad (4)$$

in which all terms are expressed in millimeters. Substituting for δ_{BD} and δ_{CE} from Eqs. (2) and (3) gives

$$\frac{\delta_A + 11.26P \times 10^{-6}}{450 + 225} = \frac{6.887P \times 10^{-6} + 11.26P \times 10^{-6}}{225}.$$

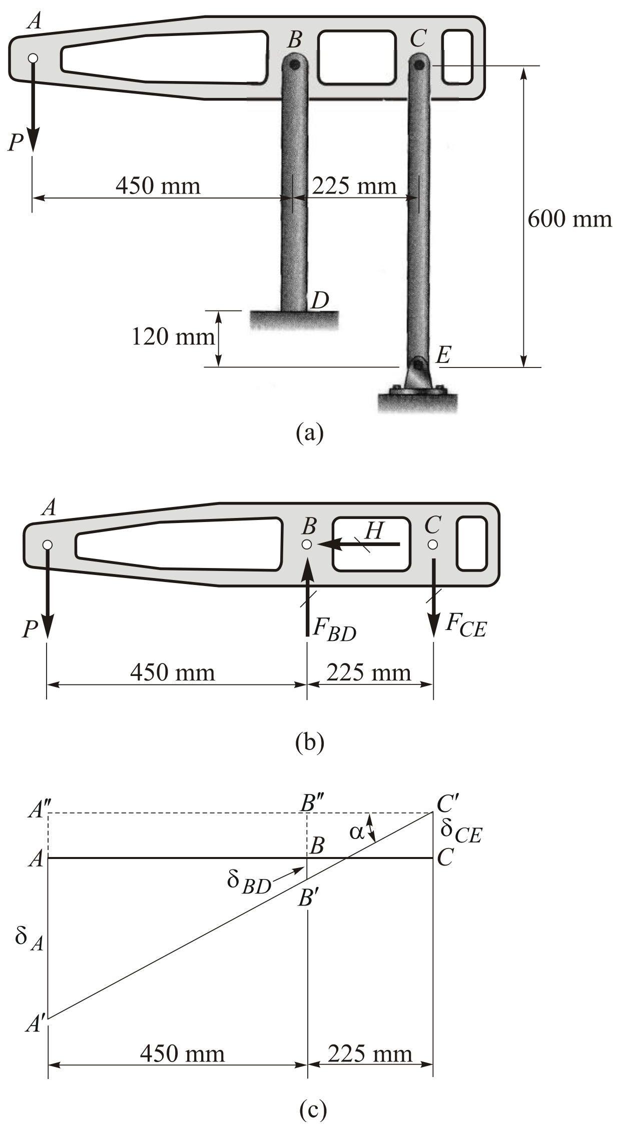


Fig. 20 Horizontal absolutely rigid beam *ABC* supported by 2 vertical bars

Finally, we substitute for δ_A its limiting value of 1.0 mm and solve the equation for the load *P*. The result is

$$P = P_{\max} = 23{,}200\ \text{N (or 23.2 kN)}.$$

When the load reaches this value, the downward displacement at point *A* is 1.0 mm.

Note 1: Since the structure behaves in a linearly elastic manner, the displacements are proportional to the magnitude of the load. For instance, if the load is one-half of $P_{\max}$, that is, if $P = 11.6\,\text{kN}$, the downward displacement of point *A* is 0.5 mm.

Note 2: To verify our assumption that line *ABC* rotates through a very small angle, we can calculate the angle of rotation α from the displacement diagram (Fig. 17c), as follows:

$$\tan\alpha = \frac{A'A''}{A''C'} = \frac{\delta_A + \delta_{CE}}{675\ \text{mm}}. \qquad (5)$$

The displacement δ_A, of point *A* is 1.0 mm, and the elongation δ_{CE} of bar *CE* is found from Eq. (3) by substituting $P = 23{,}200\,\text{N}$; the result is $\delta_{CE} = 0.261\text{mm}$. Therefore, from Eq. (5) we get

$$\tan\alpha = \frac{1.0\ \text{mm} + 0.261\ \text{mm}}{675\ \text{mm}} = \frac{1.261\ \text{mm}}{675\ \text{mm}} = 0.001868$$

from which $\alpha = 0.11^\circ$. This angle is so small that if we tried to draw the displacement diagram to scale, we would not be able to distinguish between the original line *ABC* and the rotated line $A'B'C'$. Thus, when working with displacement diagrams, we usually can consider the displacements to be very small quantities, thereby simplifying the geometry. In this example we were able to assume that points *A*, *B*, and *C* moved only vertically, whereas if the displacements were large, we would have to consider that they moved along curved paths.

6 Statically Indeterminate Rods and Rod Systems in Tension or Compression

***By a* statically determinate system *is meant a system for which all the reactions of the supports and internal force factors can be determined by means of equations of equilibrium*.**

By a* statically indeterminate system *is meant a system for which the external reactions and all the internal force factors cannot be determined by means of the method of sections and equations of equilibrium. In this case, it is necessary to create additional equations connecting displacements of points or cross-sections, taking into account that displacements are connected with internal forces by Hooke's law. These equations are named as compatibility equations. The method of opening of static indeterminacy is illustrated below.

Example 3 Singly statically indeterminate rod (see Fig. 21)

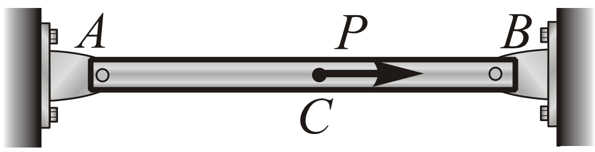


Fig. 21 The picture of statically indeterminate rod loaded by concentrated force at point *C*. *A* and *B* – immobile hinged supports

The sketch of this problem is presented on Fig. 22

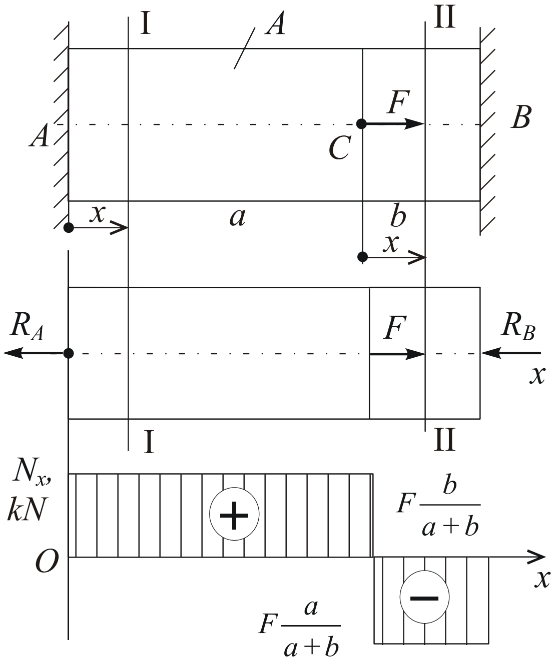


Fig. 22

Given: F, a, b. A straight homogeneous bar is rigidly fixed at the ends and subjected to a longitudinal force F.

It is necessary to determine the normal force distribution and to draw the diagram $N_x(x)$.

Note. Two reactions of supports R_A and R_B cannot be determined from one equation of equilibrium

$$\sum F_x = 0 = R_A + R_B - F\ ,$$

therefore the system is **singly statically indeterminate**.

It is necessary to have one more equation. It is the **displacement equation (syn. equation of compatibility)** which expresses the fact that the overall length of the bar remains unchanged:

$$\Delta l_{AB} = \Delta l_{AC} + \Delta l_{CB} = \Delta l^{I} + \Delta l^{II} = 0\,,$$

where

$$\Delta l^{I} = \frac{N_x^{I} a}{EA}, \qquad \Delta l^{II} = \frac{N_x^{II} b}{EA}\,.$$

In Fig. 22 it is seen that

$$N_x^{I} = R_A\,, \quad N_x^{II} = R_A - F\,.$$

Then

$$\frac{R_A a}{EA} + \frac{(R_A - F)b}{EA} = 0 \to R_A = F\frac{b}{a+b}, \quad R_B = F\frac{a}{a+b}\,,$$

$$N_x^{I}(x) = F\frac{b}{a+b}\,, \quad N_x^{II} = -F\frac{a}{a+b}\,,$$

i.e. the static indeterminacy is opened. Corresponding graph of normal force distribution is shown in Fig. 22.

Example 4 Problem of thermal stresses

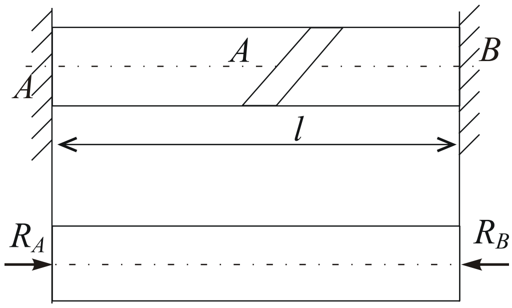


Fig. 23

Given: l, A, α_t, E, Δt°, where α_t is linear coefficient of material thermal expansion, Δt° is change in the rod temperature.

It is necessary to determine internal force N_x in the rod when it is heated by Δt°.

(1) The equation of equilibrium

$$\sum F_x = 0 \rightarrow R_A - R_B = 0, \quad R_A = R_B;$$

(2) The displacement equation must express the fact that the length of the bar remains unchanged:

$$\Delta l = \Delta l(N_x, \Delta t) = \Delta l(N_x) + \Delta l(\Delta t) = 0.$$

Note, that in deriving of this equation we used the **superposition principle**.

In this equation, $\Delta l\left(N_x\right) = \frac{N_x l}{EA} = -\frac{R_A l}{EA}$, because $N_x = -R_A$.

$$\Delta l\left(\Delta t\right) = \alpha_t \Delta t^\circ l.$$

Then

$$-\frac{R_A l}{EA} + \alpha_t \Delta t^\circ l = 0 \rightarrow R_A = EA\alpha_t \Delta t^\circ.$$

After that the diagram $N_x\left(x\right)$ may be constructed easy. It is evident that all cross-sections of the rod will be compressed.

Example 5 Design problem for statically indeterminate rod

Given: $F_1 = 10\,\text{kN}$, $F_2 = 40\,\text{kN}$, $F_3 = 60\,\text{kN}$, $a = 3\,\text{m}$, $b = 4\,\text{m}$, $c = 5\,\text{m}$, $E = 2\times10^{11}\,\text{Pa}$, $\left[\sigma\right]_t = 160\,\text{MPa}$, $\left[\sigma\right]_c = 200\,\text{MPa}$.

R.D.: 1) open static indeterminacy and design the graph of normal forces;

2) calculate cross-sectional area A;

3) design the graph of acting stresses;

4) design the graph of rod cross-section displacements.

Solution

1) Calculation of normal forces in cross-sections of the rod.

R_A, R_B are unknown reactions. It is impossible to apply the method of sections and determine normal forces without their preliminary calculation. For this, we will use:

a) equation of equilibrium: $\sum F_x = 0 = R_A - F_1 + F_3 - F_2 + R_B$;

b) equation of segments elongations compatibility: $\Delta l_{AB} = 0$ or

$$\Delta l_{AC} + \Delta l_{CD} + \Delta l_{DE} + \Delta l_{EB} = 0. \tag{1}$$

Since $N_x^I(x) = +R_A$, $N_x^{II}(x) = +R_A - F_1$, $N_x^{III}(x) = +R_A - F_1 + F_3$, $N_x^{IV}(x) = +R_A - F_1 + F_3 - F_2$ let us rewrite the compatibility equation (1) using the equation of Hooke's law:

$$\Delta l = \frac{N_x l}{EA}. \tag{2}$$

Equation (1) becomes

$$\frac{R_A c}{3EA} + \frac{(R_A - F_1)b/3}{2EA} + \frac{(R_A - F_1 + F_3)2b/3}{2EA} + \frac{(R_A - F_1 + F_3 - F_2)a}{EA} = 0. \tag{3}$$

After simplifying $R_A = -13.5\,\text{kN}$, i.e. originally downdirected R_A must be changed on opposite. After this, statical indeterminacy is opened since it is possible to calculate normal forces in the portions:

I–I $0 < x < c$

$$N_x^I(x) = -R_{Aact} = -13.5\,\text{kN};$$

II–II $0 < x < b/3$

$$N_x^{II}(x) = -R_{Aact} - F_1 = -13.5 - 10 = -23.5\,\text{kN};$$

III–III $0 < x < 2b/3$

$$N_x^{III}(x) = -R_{Aact} - F_1 + F_3 = -13.5 - 10 + 60 = +36.5\,\text{kN};$$

IV–IV $0 < x < a$

$$N_x^{IV}(x) = -R_{Aact} - F_1 + F_3 - F_2 = -13.5 - 10 + 60 - 40 = -3.5\,\text{kN}.$$

2) Calculation of cross-sectional area A from conditions of strength.

Note, that since ultimate strength of the rod material is different in tension and compression, we will write the conditions of strength for both most tensile and most compressed portions.

a) for one tensile portion

$$\sigma_{\max}^{tens} = \sigma_{\max}^{III} = \frac{N_x^{III}}{2A_t} \le [\sigma]_t \to A_t = \frac{N_x^{III}}{2[\sigma]_t} = \frac{36.5\times10^3}{2\times160\times10^6} = 1.14\times10^{-4}\,\text{m}^2;$$

b) since there are three compressed portions in this rod it is necessary to determine preliminary the most compressed one comparing two relations

$$\sigma^{II} = \frac{N_x^{II}}{2A} = \frac{23.5\times10^3}{2A} = \frac{11.75\times10^3}{A}$$

and

$$\sigma^{IV} = \frac{N_x^{IV}}{A} = \frac{3.5\times10^3}{A}$$

we found that second portion is critical in compression since

$$\frac{11.75\times10^3}{A} > \frac{3.5\times10^3}{A}.$$

In result, for the most compressed part of the rod

$$\sigma_{\max}^{comp} = \sigma_{\max}^{II} = \frac{N_x^{II}}{2A_c} \le [\sigma]_c \to A_c = \frac{N_x^{II}}{2[\sigma]_c} = \frac{23.5\times10^3}{2\times200\times10^{-4}} = 0.588\times10^{-4}\,\text{m}^2.$$

Comparing two values $A_c = 0.588\times10^{-4}\,\text{m}^2$ and $A_t = 1.14\times10^{-4}\,\text{m}^2$ we select the larger one: $A = 1.14\times10^{-4}\,\text{m}^2$.

3) Calculation in actual stresses in the portions of the rod:

$$\sigma_x^{I} = \frac{N_x^{I}}{3A} = -\frac{13.5\times10^3}{3\times1.14\times10^{-4}} = -39.47\,\text{MPa} < [\sigma]_c,$$

$$\sigma_x^{II} = \frac{N_x^{II}}{2A} = -\frac{23.5\times10^3}{2\times1.14\times10^{-4}} = -103.07\,\text{MPa} < [\sigma]_c,$$

$$\sigma_x^{III} = \frac{N_x^{III}}{2A} = +\frac{36.5\times10^3}{2\times1.14\times10^{-4}} = +160\,\text{MPa} = [\sigma]_t,$$

$$\sigma_x^{IV} = \frac{N_x^{IV}}{A} = -\frac{3.5\times10^3}{1.14\times10^{-4}} = -30.7\,\text{MPa} < [\sigma]_c.$$

The results show that the rod is strong since all its parts are strong.

4) Determination of the rod elongations.

To solve this problem, we will use the following formula:

$\Delta l(x) = \frac{N_x x}{EA} = kx$ i.e. the elongation is the binear function of the segment length

Let us begin determining the displacements of the portions boundaries (points C, D, E) calculating corresponding segments elongations. In this, we will use point A as the origin. Corresponding displacements are

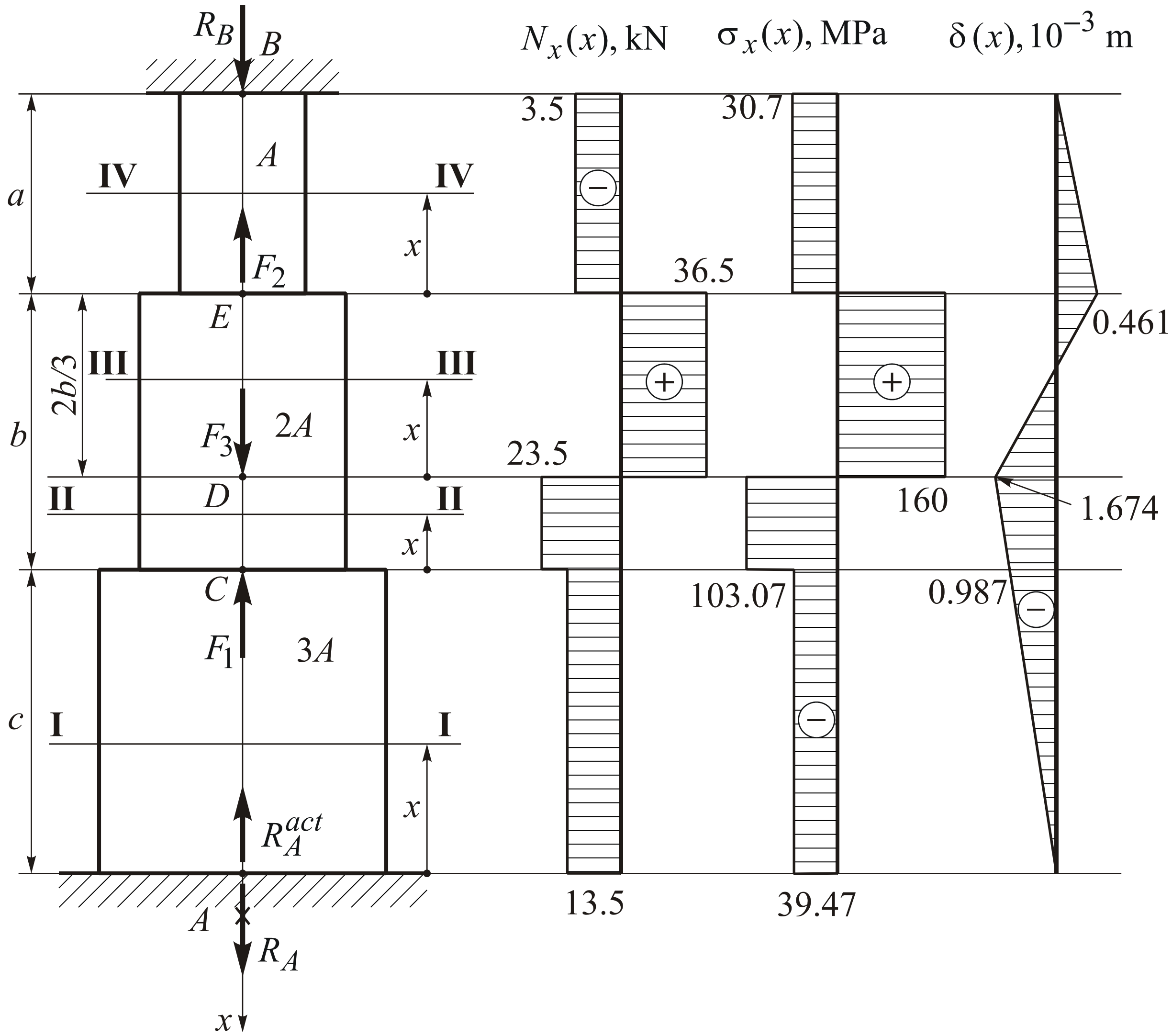


Fig. 23

$$\delta_{p.C} = \Delta l_{AC} = \frac{N_x^I c}{3EA} = -\frac{13.5\times10^3\times5}{3\times2\times10^{11}\times1.14\times10^{-4}} = -0.987\times10^{-3}\,\text{m},$$

$$\delta_{p.D} = \Delta l_{AD} = \Delta l_{AC} + \Delta l_{CD} = -0.987\times10^{-3} + \frac{N_x^{II} b}{3\times2EA} = -0.987\times10^{-3} -$$

$$-\frac{23.5\times10^3\times4}{3\times2\times2\times10^{11}\times1.14\times10^{-4}} = --0.987\times10^{-3} - 0.687\times10^{-3} = -1.674\times10^{-3}\,\text{m},$$

$$\delta_{p.E} = \Delta l_{AE} = \Delta l_{AD} + \Delta l_{DE} = -1.674\times10^{-3} + \frac{N_x^{III}\,2b}{3\times2EA} = -1.674\times10^{-3} +$$

$$+\frac{36.5\times10^3\times2\times4}{3\times2\times2\times10^{11}\times1.14\times10^{-4}} = -1.674\times10^{-3} + 2.135\times10^3 = +0.461\times10^{-3}\,\text{m},$$

$$\delta_{p.B} = \Delta l_{AB} = \Delta l_{AE} + \Delta l_{EB} = +0.461\times10^{-3} + \frac{N_x^{IV} a}{EA} = -+0.461\times10^{-3} -$$

$$-\frac{3.5\times10^3\times3}{2\times10^{11}\times1.14\times10^{-4}} = +0.461\times10^{-3} - 0.461\times10^{-3} = 0\,.$$

Note, that the elongation of *AB*-segment is 0 since it is fixed between two absolutely rigid supports.

Example 6 Problem of assembly stresses

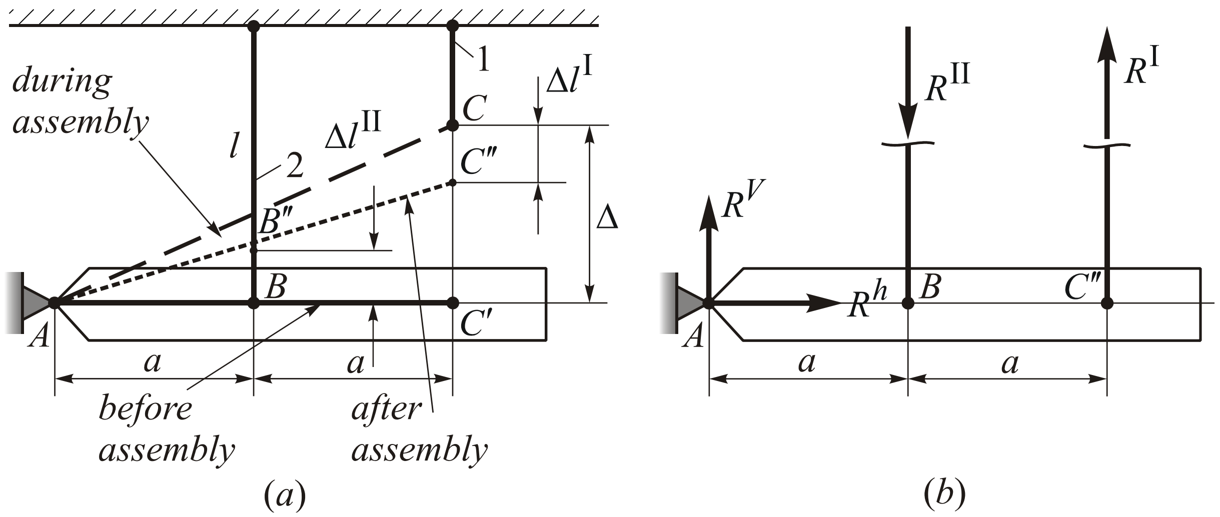


Fig. 24

When assembling a bar system (Fig. 24) it was found that there were inaccuracies in the length of the bars (Δ is inaccuracy in the length of the first bar).

During assembly, the bar (1) was put into place by connecting the hinges *C* and *C′* to one point.

It is necessary to determine the forces in the bars after assembly, i.e. when the bar axis becomes equal to *AC″*.

We have four unknown forces R^I, R^{II}, R^v, R^h and three equations of equilibrium, which are not enough for determining internal forces in two elastic supporting rods (R^I, R^{II}). Consequently, the system is singly statically indeterminate.

Suppose that after assembly the hinge *C* has moved downward through a distance Δl^I (real elongation of the first rod) and took position *C″*, and the hinge *B* has moved upward to *B″* (*B‴B* is really shortening of this rod Δl^{II}).

From the condition of equilibrium

$$\sum M_A = 0: \quad R^I 2a - R^{II} a = 0, \quad R^{II} = 2R^I .$$

Compatibility equation is

$$\frac{\Delta - |\Delta l^I|}{|\Delta l^{II}|} = \frac{2a}{a} = 2, \qquad (*)$$

where

$$\Delta l^I = \frac{N^I l^I}{EA}; \quad \Delta l^{II} = \frac{N^{II} l^{II}}{EA}, \quad N^I = +R^I, \quad N^{II} = -R^{II} .$$

Then

$$\frac{\Delta - \dfrac{R^I l}{EA}}{\dfrac{R^{II} l}{EA}} = 2, \quad \frac{\Delta EA - R^I l}{R^{II} l} = 2, \quad \Delta EA - R^I l = 2R^{II} l = 4R^I l;$$

$$R^I = \frac{1}{5}\frac{\Delta EA}{l}, \quad R^{II} = \frac{2}{5}\frac{\Delta EA}{l};$$

$$N_x^I(x) = R^I, \quad N_x^{II}(x) = -R^{II} .$$

It should be noted that relation (*) connects moduli of the displacements.

7 Examples of engineering problems solution

Example 7 Stress analysis of tubular post with round core

A solid circular steel cylinder *S* is encased in a hollow circular copper tube *C* (Figs. 25a and 25b). The cylinder and tube are compressed between the rigid plates of a testing machine by compressive forces *P*. The steel cylinder has cross-sectional area A_s and modulus of elasticity E_s, the copper tube has area A_c and modulus E_c, and both parts have length *L*.

Determine the following quantities: (a) the compressive normal forces N_s in the steel cylinder and N_c in the copper tube; (b) the corresponding compressive stresses σ_s and σ_c and (c) the shortening δ of the assembly.

Solution (a) *Compressive forces in the steel cylinder and copper tube.* We begin by removing the upper plate of the assembly in order to expose the compressive forces P_s and P_c acting on the steel cylinder and copper tube, respectively (Fig. 25c). The normal force $N_x = P_s$ is the resultant of the uniformly distributed stresses acting over the cross section of the steel cylinder, and the normal force $N_c = P_c$ is the resultant of the stresses acting over the cross section of the copper tube.

Equation of equilibrium. A free-body diagram of the upper plate is shown in Fig. 22d. This plate is subjected to the force *P* and to the unknown compressive forces P_s and P_c; thus, the equation of equilibrium is

$$\sum F_{vert} = 0, \quad P_s + P_c - P = 0. \tag{1}$$

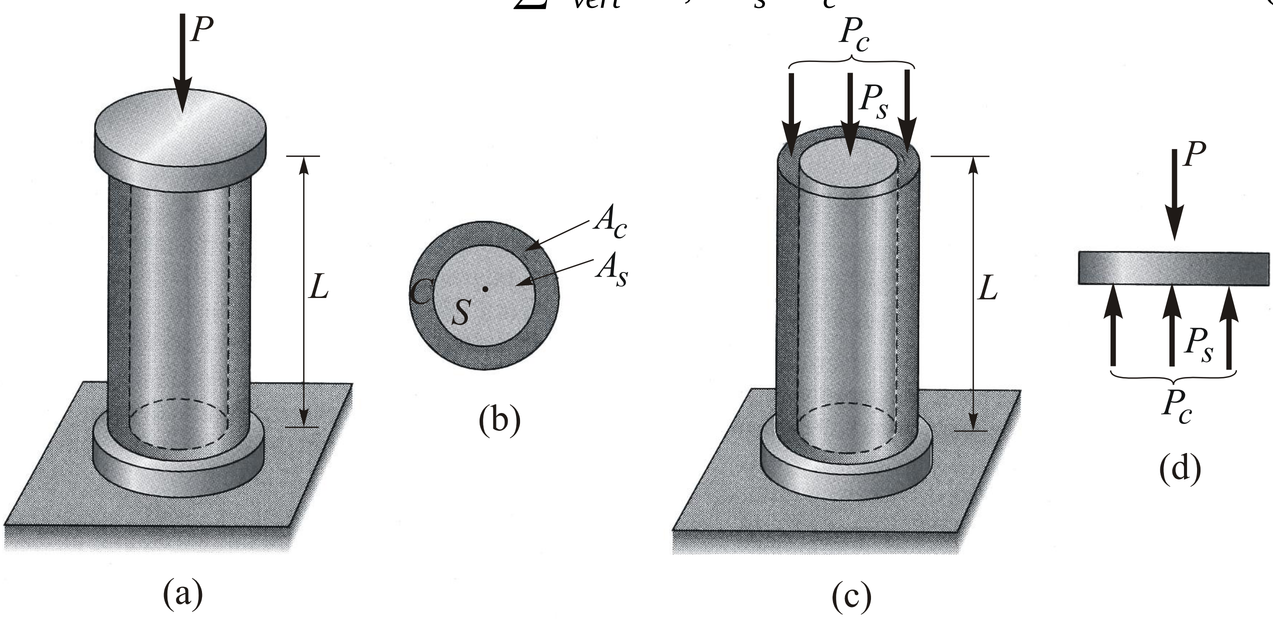


Fig. 25 Analysis of a statically indeterminate structure

This equation, which is the only nontrivial equilibrium equation available, contains two unknowns. Therefore, we conclude that the structure is statically indeterminate.

Equation of compatibility. Because the end plates are rigid, the steel cylinder and copper tube must shorten by the same amount. Denoting the shortenings of the steel and copper parts by δ_S and δ_C, respectively, we obtain the following equation of compatibility:

$$\delta_S = \delta_C \tag{2}$$

Force-displacement relations. The changes in lengths of the cylinder and tube can be obtained from the general equation $\delta = N_x L / EA$. Therefore, in this example the force-displacement relations are

$$\delta_S = \frac{P_S L}{E_S A_S}, \quad \delta_C = \frac{P_C L}{E_C A_C} \quad \text{since} \quad P_S = N_S \quad \text{and} \quad P_C = N_C. \tag{3, 4}$$

Substituting these relations into the equation of compatibility (Eq. 2) gives

$$\frac{P_S}{E_S A_S} = \frac{P_C}{E_C A_C} \tag{5}$$

This equation gives the compatibility condition in terms of the unknown forces.

Solution of equations. We now solve simultaneously the equations of equilibrium and compatibility (Eqs. 1 and 5) and obtain the axial forces in the steel cylinder and copper tube:

$$P_S = P\left(\frac{E_S A_S}{E_S A_S + E_C A_C}\right), \quad P_C = P\left(\frac{E_C A_C}{E_S A_S + E_C A_C}\right). \tag{6, 7}$$

These equations show that the compressive forces in the steel and copper parts are directly proportional to their respective axial rigidities and inversely proportional to the sum of their rigidities.

(b) *Compressive stresses in the steel cylinder and copper tube.* Knowing the axial internal forces, we can now obtain the compressive stresses in the two materials:

$$\sigma_s = \frac{N_s}{A_s} = \frac{PE_s}{E_s A_s + E_c A_c}, \quad \sigma_c = \frac{N_c}{A_c} = \frac{PE_c}{E_s A_s + E_c A_c}. \tag{8, 9}$$

Note that the stresses are proportional to the moduli of elasticity of the respective materials. Therefore, the "stiffer" material has the larger stress.

(c) *Shortening of the assembly*. The shortening δ of the entire assembly can be obtained from either Eq. (3) or Eq. (4). Thus, upon substituting the forces (from Eqs. 6 and 7), we get

$$\delta = \frac{P_s L}{E_s A_s} = \frac{P_c L}{E_c A_c} = \frac{PL}{E_s A_s + E_c A_c} \tag{10}$$

This result shows that the shortening of the assembly is equal to the total load divided by the sum of the stiffness of the two parts.

Example 8 Analysis of a statically indeterminate structure with absolutely rigid element

A horizontal rigid bar *AB* is pinned at end *A* and supported by two wires (*CD* and *EF*) at points *D* and *F* (Fig. 26a). A vertical load *P* acts at end *B* of the bar. The bar has length $3b$ and wires *CD* and *EF* have lengths L_1 and L_2, respectively. Also, wire *CD* has diameter d_1 and modulus of elasticity E_1, wire *EF* has diameter d_2 and modulus E_2.

(a) Obtain formulas for the allowable load *P* if the allowable stresses in wires *CD* and *EF*, respectively, are σ_1 and σ_2. (Disregard the weight of the bar itself).

(b) Calculate the allowable load *P* for the following conditions: wire *CD* is made of aluminum with modulus $E_1 = 72$ GPa, diameter $d_1 = 4.0$ mm, and length $L_1 = 0.40$ m. Wire *EF* is made of magnesium with modulus $E_2 = 45$ GPa, diameter $d_2 = 3.0$ mm, and length $L_2 = 0.30$ m. The allowable stresses in the aluminum and magnesium wires are $\sigma_1 = 200$ MPa and $\sigma_2 = 175$ MPa, respectively.

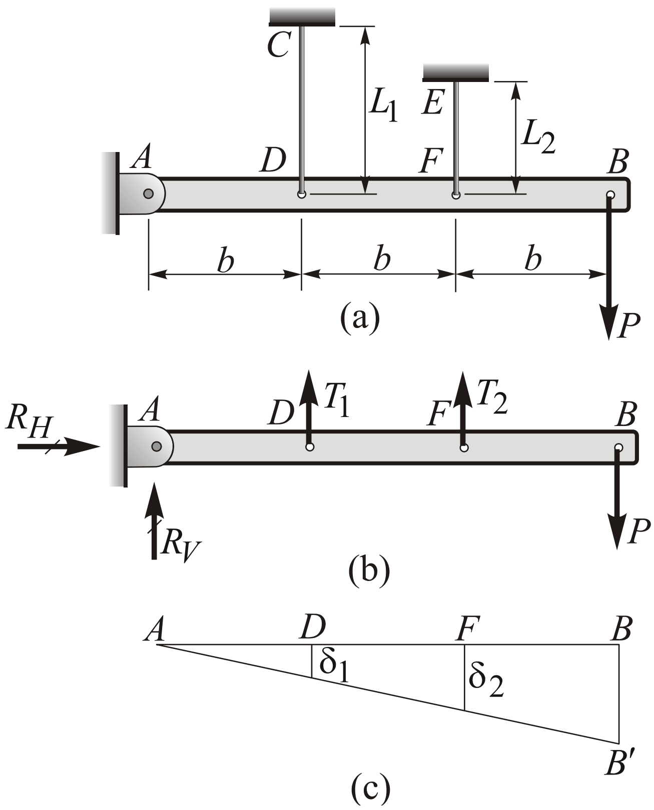


Fig. 26 Analysis of a statically indeterminate structure

Solution *Equations of equilibrium.* We begin the analysis by drawing a free-body diagram of bar *AB* (Fig. 26b). In this diagram T_1 and T_2 are the unknown tensile forces in the wires and R_H and R_V are the horizontal and vertical components of the reaction at the support. The structure is statically indeterminate because there are four unknown forces (T_1, T_2, R_H, and R_V) but only three independent equations of equilibrium. Taking moments about point *A* (with counterclockwise moments being positive) yields

$$\sum M_A = 0,$$

$$T_1 b + T_2(2b) - P(3b) = 0 \quad \text{or} \quad T_1 + 2T_2 = 3P. \qquad (1)$$

The other two equations, obtained by summing forces in the horizontal direction and summing forces in the vertical direction, are of no benefit in finding T_1 and T_2.

Equation of compatibility. To obtain an equation pertaining to the displacements, we observe that the load *P* causes bar *AB* to rotate about the pin support at *A*, thereby stretching the wires. The resulting displacements are shown in the displacement diagram of Fig. 26c, where line *AB* represents the original position of the rigid bar and line *AB′* represents the rotated position. The displacements δ_1 and δ_2 are the elongations of the wires. Because these displacements are very small, the bar rotates through a very small angle (shown highly exaggerated in the figure) and we can make calculations on the assumption that points *D*, *F*, and *B* move vertically downward (instead of moving along the arcs of circles).

Because the horizontal distances *AD* and *DF* are equal, we obtain the following geometric relationship between the elongations:

$$\delta_2 = 2\delta_1. \qquad (2)$$

Equation (2) is the equation of compatibility.

Force-displacement relations. Since the wires behave in a linearly elastic manner, their elongations can be expressed in terms of the unknown forces T_1 and T_2 by means of the following expressions:

$$\delta_1 = \frac{T_1 L_1}{E_1 A_1}, \quad \delta_2 = \frac{T_2 L_2}{E_2 A_2},$$

in which A_1 and A_2 are the cross-sectional areas of wires *CD* and *EF*, respectively; that is,

$$A_1 = \frac{\pi d_1^2}{4}, \quad A_2 = \frac{\pi d_2^2}{4}.$$

For convenience in writing equations, let us introduce the following notation for the flexibilities of the wires:

$$f_1 = \frac{L_1}{E_1 A_1}, \qquad f_2 = \frac{L_2}{E_2 A_2}. \qquad (3, 4)$$

Then the force-displacement relations become

$$\delta_1 = f_1 T_1, \quad \delta_2 = f_2 T_2. \qquad (5)$$

Substituting these expressions into the equation of compatibility (Eq. 1) gives

$$f_2T_2 = 2f_1T_1. \qquad (6)$$

We now have expressed the equation of compatibility in terms of the unknown forces.

Solution of equations. The equation of equilibrium (Eq. 1) and the equation of compatibility (Eq. 6) each contain the forces T_1 and T_2 as unknown quantities. Solving the two equations simultaneously yields

$$T_1 = \frac{3f_2P}{4f_1 + f_2}, \quad T_2 = \frac{6f_1P}{4f_1 + f_2}. \qquad (7, 8)$$

Knowing the forces T_1 and T_2, we can easily find the elongations of the wires from the force-displacement relations.

(a) *Allowable load P.* Now that the statically indeterminate analysis is completed and the forces in the wires are known, we can determine the permissible value of the load *P*. The stress σ_1 in wire *CD* and the stress σ_2 in wire *EF* are readily obtained from the forces (Eqs. 7 and 8):

$$\sigma_1 = \frac{T_1}{A_1} = \frac{3P}{A_1}\left(\frac{f_2}{4f_1 + f_2}\right), \quad \sigma_2 = \frac{T_2}{A_2} = \frac{6P}{A_2}\left(\frac{f_1}{4f_1 + f_2}\right). \qquad (9, 10)$$

From the first of these equations we solve for the permissible force P_1, based upon the allowable stress σ_1 in the aluminum wire:

$$P_1 = \frac{\sigma_1A_1(4f_1 + f_2)}{3f_2}. \qquad (11)$$

Similarly, from the second equation we get the permissible force P_2 based upon the allowable stress σ_2 in the magnesium wire:

$$P_2 = \frac{\sigma_2A_2(4f_1 + f_2)}{6f_1}. \qquad (12)$$

The smaller of these two values of the load is the maximum allowable load $[P]$.

(b) *Numerical calculations for the allowable load.* Using the given data and the preceding equations, we obtain following numerical values:

$$A_1 = \frac{\pi d_1^2}{4} = \frac{\pi\left(4.0\text{mm}\right)^2}{4} = 12.57\text{mm}^2,$$

$$A_2 = \frac{\pi d_2^2}{4} = \frac{\pi\left(3.0\text{mm}\right)^2}{4} = 7.069\text{mm}^2,$$

$$f_1 = \frac{L_1}{E_1 A_1} = \frac{0.40\text{m}}{(72\text{GPa})(12.57\text{mm}^2)} = 0.4420\times10^{-6}\text{m/N},$$

$$f_2 = \frac{L_2}{E_2 A_2} = \frac{0.30\text{m}}{(45\text{GPa})(7.069\text{mm}^2)} = 0.9431\times10^{-6}\text{m/N}.$$

Also, the allowable stresses are

$$\sigma_1 = 200\,\text{MPa}, \quad \sigma_2 = 175\,\text{MPa}.$$

Therefore, substituting into Eqs. (11) and (12) gives

$$P_1 = 2.41\text{kN}, \quad P_2 = 1.26\,\text{kN}.$$

The first result is based upon the allowable stress σ_1 in the aluminum wire and the second is based upon the allowable stress σ_2 the magnesium wire. The allowable load is the smaller of the two values:

$$[P] = 1.26\,\text{kN}.$$

At this load the stress in the magnesium is 175 MPa (the allowable stress) and the stress in the aluminum is $(1.26/2.41)(200\,\text{MPa}) = 105\,\text{MPa}$. As expected, this stress is less than the allowable stress of 200 MPa.

Example 9 Calculation of internal forces in statically indeterminate rod system with absolutely rigid element

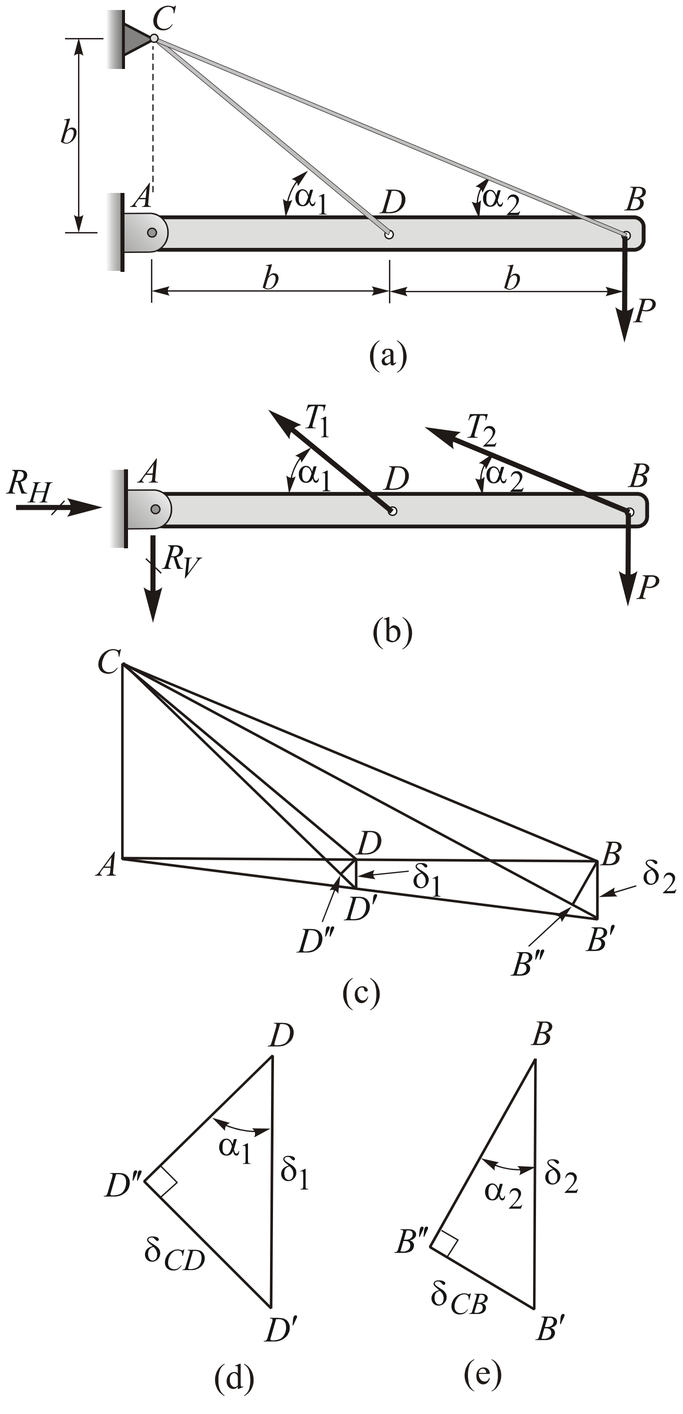


Fig. 27 Analysis of a statically indeterminate structure

A horizontal rigid bar *ADB* of length $2b$ is pinned to a support at *A* and held by two inclined wires *CD* and *CB* (Fig. 27a). The wires are attached to a support at point *C*, which is located at a vertical distance *b* above point *A*. Both wires are made of the same material and have the same diameter.

Determine the tensile forces T_1 and T_2 in wires *CD* and *CB*, respectively, due to the vertical load *P* acting at the end of the bar.

Solution *Equation of equilibrium.* A free-body diagram of bar *AB* (Fig. 24b) shows that there are four unknown forces, namely, the tensile forces T_1 and T_2 in the wires and the two reaction components (R_H and R_V) at the pin support. Since there are only three independent equations of equilibrium, the structure is singly statically indeterminate. We can obtain an equation of equilibrium containing only T_1 and T_2 as unknowns by summing moments about point *A*:

$$\sum M_A = 0,$$

$$bT_1 \sin\alpha_1 + 2bT_2 \sin\alpha_2 - 2bP = 0 \tag{1}$$

in which α_1 and α_2 are the angles between the wires and the bar.

From the geometry of the structure (Fig. 27a) we see that

$$\sin\alpha_1 = \frac{CA}{CD} = \frac{b}{b\sqrt{2}} = \frac{1}{\sqrt{2}},$$

$$\sin\alpha_2 = \frac{CA}{CB} = \frac{b}{b\sqrt{5}} = \frac{1}{\sqrt{5}}. \tag{2}$$

Substituting these values into Eq. (1) and rearranging, we get

$$\frac{T_1}{\sqrt{2}} + \frac{2T_2}{\sqrt{5}} = 2P \tag{3}$$

which is the final form of the equation of equilibrium.

Equation of compatibility. An equation of compatibility can be obtained by considering the displacements of points *D* and *B*. For this purpose, we draw the displacement diagram shown in Fig. 27c. Line *AB* represents the original position of bar *AB* and line *AB'* represents the rotated position. Since the angle of rotation of bar *AB* is very small, the displacements δ_1 and δ_2 of points *D* and *B*, respectively, can be treated as small vertical displacements. Because the distance from *A* to *B* is twice the distance from *A* to *D*, we see that

$$\delta_2 = 2\delta_1 \tag{4}$$

which is the equation of compatibility.

Force-displacement relations. Now we need to relate the forces T_1 and T_2 in the wires to the vertical displacements δ_1 and δ_2 at points *D* and *B*, respectively. Let us begin by considering wire *CD*, which rotates from its original position *CD* to its final position *CD'* (Fig. 27c). We draw a perpendicular *DD''* from point *D* to line *CD'*. Because the displacements and angle changes are very small, we can assume that the distance *CD''* is equal in length to the distance *CD*. (That is, a circular arc *DD''* with point *C* as its center is indistinguishable from the perpendicular line *DD''*.)

The displacement triangle *DD'D''* in Fig. 27c is redrawn for clarity in Fig. 27d. The distance *DD'* is the vertical displacement δ_1 of point *D* and the distance *D'D''* is the elongation δ_{CD} of wire *CD*. The angle α_1 appears in the triangle as angle *DD'D''*. To prove this, we note that line *DD'* is perpendicular to line *AB*, and line *DD''* is perpendicular to line *CD* (for small angles of rotation). Therefore, from the displacement triangle *DD'D''* we get

$$\delta_{CD} = \delta_1 \sin\alpha_1 = \frac{\delta_1}{\sqrt{2}}. \tag{5}$$

This equation provides the geometric relation between the elongation of wire *CD* and the downward displacement of point *D*.

The elongation δ_{CD} of wire *CD* is related to the force in the wire by the following force-displacement relation:

$$\delta_{CD} = \frac{T_1 L_{CD}}{EA} = \frac{T_1 b\sqrt{2}}{EA} \tag{6}$$

in which $L_{CD} = b\sqrt{2}$, is the length of wire *CD*. Combining the last two equations for δ_{CD}, we get

$$\delta_1 = \frac{2bT_1}{EA}. \tag{7}$$

In a similar manner, we can relate the displacement δ_2 to the elongation δ_{CB} of wire *CB* (see Fig. 27e) and obtain

$$\delta_2 = \frac{5bT_2}{EA}. \tag{8}$$

Equations (7) and (8) are the force-displacement relations that give the displacements of points *D* and *B* in terms of the unknown forces in the wires.

Substituting δ_1 and δ_2 from the force-displacement relations into the equation of compatibility (Eq. 4) yields

$$5T_2 = 4T_1 \tag{9}$$

which is the equation of compatibility in terms of the unknown forces.

Solution of equations. As the final step in this example we solve simultaneously the equation of equilibrium (Eq. 3) and the equation of compatibility (Eq. 9). The results are

$$T_1 = \frac{10\sqrt{10}P}{8\sqrt{2}+5\sqrt{5}} = 1.406P, \quad T_2 = \frac{4T_1}{5} = 1{,}125P. \qquad (10, 11)$$

Thus, we have found the tensile forces in the wires by solving the statically indeterminate structure.

Knowing the tensile forces in the wires, we can easily determine all other force and displacement quantities. For instance, the reactions at support *A* can be found from equilibrium equations and the displacements δ_1 and δ_2 can be found from Eqs. (7) and (8).

Note: In this example, the final solution for the forces does not involve the axial rigidity *EA* of the wires. (This fact is readily apparent in a symbolic solution but might escape notice in a numerical solution.) The reason is that both wires have the same axial rigidity, and therefore *EA* cancels out of the solution. If each wire had a different rigidity, the rigidities would appear in the final expressions.

Example 10 Thermal effects in statically indeterminate rod systems

A sleeve in the form of a circular tube of length *L* is placed around a bolt and fitted between washers at each end (Fig. 28a). The nut is then turned until it is just snug. The sleeve and bolt are made of different materials and have different cross-sectional areas. (a) If the temperature of the entire assembly is raised by an amount ΔT, what stresses σ_s and σ_b are developed in the sleeve and bolt, respectively? (b) What is the increase δ in the length *L* of the sleeve and bolt? (Assume that the coefficient of thermal expansion α_s of the sleeve is greater than the coefficient α_b of the bolt).

Solution Because the sleeve and bolt are of different materials, they will elongate by different amounts when heated and allowed to expand freely. However, when they are held together by the assembly, free expansion cannot occur and thermal stresses are developed in both materials. To find these stresses, we use the same concepts as in any statically indeterminate analysis – equilibrium equations, compatibility equations, and displacement relations. However, we cannot formulate these equations until we disassemble the structure.

A simple way to cut the structure is to remove the head of the bolt, thereby allowing the sleeve and bolt to expand freely under the temperature change ΔT (Fig. 28b). The resulting elongations of the sleeve and bolt are denoted δ_1 and δ_2, respectively, and the corresponding **temperature-displacement relations** are

$$\delta_1 = \alpha_s(\Delta T)L, \quad \delta_2 = \alpha_b(\Delta T)L. \tag{1}$$

Since α_s is greater than α_b, the elongation δ_1 is greater than δ_2, as shown in Fig. 28b.

The axial forces in the sleeve and bolt must be such that they shorten the sleeve and stretch the bolt until the final lengths of the sleeve and bolt are the same. These forces are shown in Fig. 28c, where P_s denotes the compressive force in the sleeve and P_b denotes the tensile force in the bolt. The corresponding shortening δ_3 of the sleeve and elongation δ_4 of the bolt are

$$\delta_3 = \frac{P_s L}{E_s A_s}, \quad \delta_4 = \frac{P_b L}{E_b A_b} \tag{2}$$

in which $E_s A_s$ and $E_b A_b$ are the respective axial rigidities. Equations (2) are the **load-displacement relations**.

Now we can write an **equation of compatibility** expressing the fact that the final elongation δ is the same for both the sleeve and the bolt. The elongation of the sleeve is $\delta_1 - \delta_3$ and of the bolt is $\delta_2 + \delta_4$; therefore,

$$\delta = \delta_1 - \delta_3 = \delta_2 + \delta_4. \tag{3}$$

Substituting the temperature-displacement and load-displacement relations (Eqs. 1 and 2) into this equation gives

$$\delta = \alpha_s(\Delta T)L - \frac{P_s L}{E_s A_s} = \alpha_b(\Delta T)L + \frac{P_b L}{E_b A_b} \tag{4}$$

from which we get

$$\frac{P_s L}{E_s A_s} + \frac{P_b L}{E_b A_b} = \alpha_s (\Delta T) L - \alpha_b (\Delta T) L \,, \qquad (5)$$

which is the final form of the compatibility equation. Note that it contains the forces P_s and P_b as unknowns.

An **equation of equilibrium** is obtained from Fig. 28c, which is a free-body diagram of the part of the assembly remaining after the head of the bolt is removed. Summing forces in the horizontal direction gives

$$P_s = P_b \qquad (6)$$

which expresses the obvious fact that the compressive force in the sleeve is equal to the tensile force in the bolt.

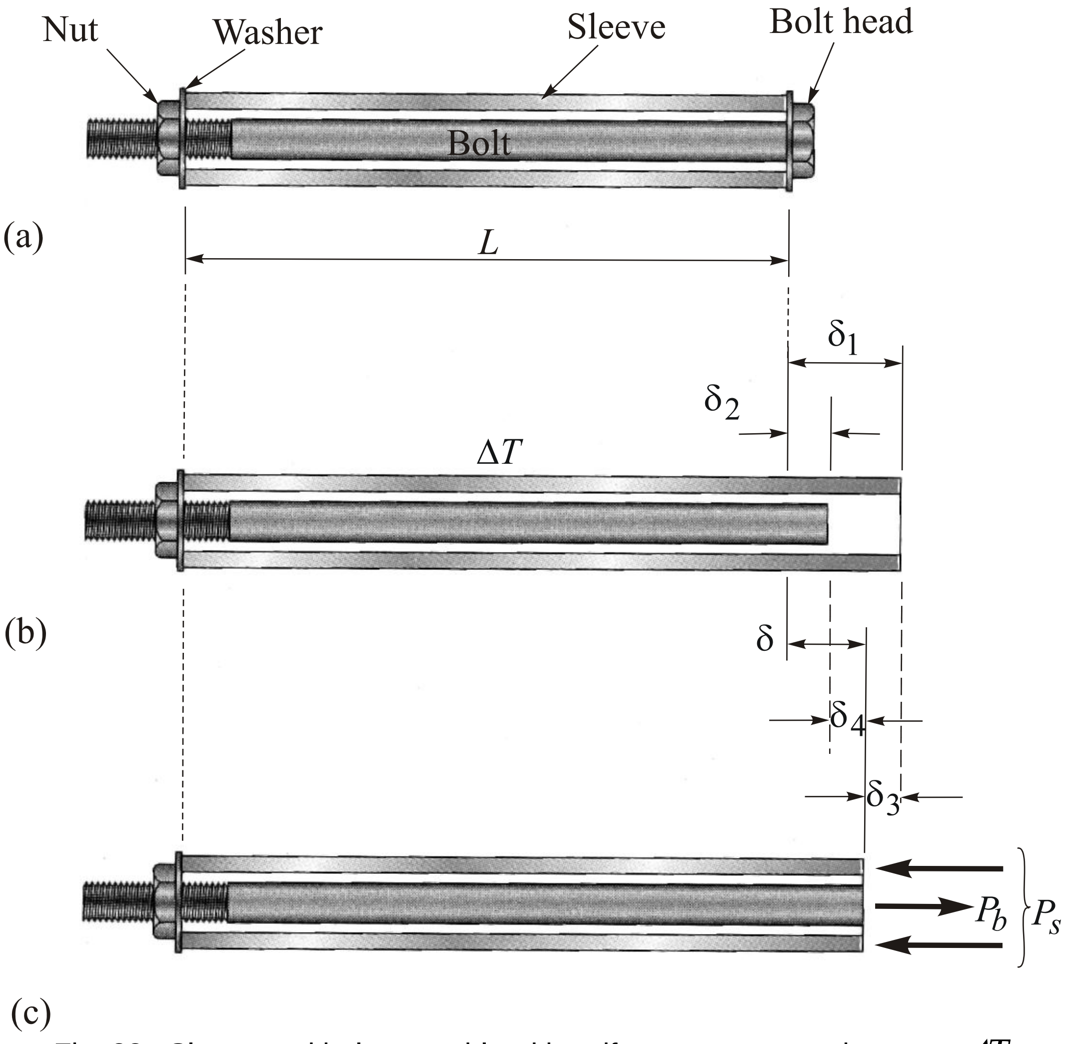


Fig. 28 Sleeve and bolt assembly with uniform temperature increase ΔT

We now solve simultaneously the equations of compatibility and equilibrium (Eqs. 5 and 6) and obtain the axial forces in the sleeve and bolt:

$$P_s = P_b = \frac{(\alpha_s - \alpha_b)(\Delta T)E_s A_s E_b A_b}{E_s A_s + E_b A_b}. \tag{7}$$

When deriving this equation, we assumed that the temperature increased and that the coefficient α_s was greater than the coefficient α_b. Under these conditions, P_s is the compressive force in the sleeve and P_b is the tensile force in the bolt.

The results will be quite different if the temperature increases but the coefficient α_s is less than the coefficient α_b. Under these conditions, a gap will open between the bolt head and the sleeve and there will be no stresses in either part of the assembly.

(a) *Stresses in the sleeve and bolt.* Expressions for the stresses σ_s and σ_b in the sleeve and bolt, respectively, are obtained by dividing the corresponding forces by the appropriate areas:

$$\sigma_s = \frac{P_s}{A_s} = \frac{(\alpha_s - \alpha_b)(\Delta T)E_s E_b A_b}{E_s A_s + E_b A_b}, \tag{8}$$

$$\sigma_b = \frac{P_b}{A_b} = \frac{(\alpha_s - \alpha_b)(\Delta T)E_s E_b A_s}{E_s A_s + E_b A_b}. \tag{9}$$

Under the assumed conditions, the stress σ_s in the sleeve is compressive and the stress σ_b in the bolt is tensile. It is interesting to note that these stresses are independent of the length of the assembly and their magnitudes are inversely proportional to their respective areas (that is, $\sigma_s / \sigma_b = A_b / A_s$).

(b) *Increase in length of the sleeve and bolt.* The elongation δ of the assembly can be found by substituting either P_s or P_b from Eq. (7) into Eq. (4), yielding

$$\delta = \frac{(\alpha_s E_s A_s + \alpha_b E_b A_b)(\Delta T)L}{E_s A_s + E_b A_b}. \tag{10}$$

With the preceding formulas available, we can readily calculate the forces, stresses, and displacements of the assembly for any given set of numerical data. **Note:** As a partial check on the results, we can see if Eqs. (7), (8), and (10) reduce to known values in simplified cases. For instance, suppose that the bolt is rigid and therefore unaffected by temperature changes. We can represent this situation by setting $\alpha_b = 0$ and letting E_b become infinitely large, thereby creating an assembly in which the sleeve is held between rigid supports. Substituting the stated values of α_b and E_b into Eqs. (7), (8), and (10), we find

$$P_s = E_s A_s \alpha_s (\Delta T), \quad \sigma_s = E_s \alpha_s (\Delta T), \quad \delta = 0.$$

As a second special case, suppose that the sleeve and bolt are made of the same material. Then both parts will expand freely and will lengthen the same amount when the temperature changes. No forces or stresses will be developed. To see if the derived equations predict this behavior, we will assume that both parts have properties α, E, and A. Substituting these values into Eqs. (7 – 10) we get

$$P_s = P_b = 0, \quad \sigma_s = \sigma_b = 0, \quad \delta = \alpha(\Delta T)L,$$

which are the expected results.

As a third special case, suppose we remove the sleeve so that only the bolt remains. The bolt is then free to expand and no forces or stresses will develop. We can represent this case by setting $A_s = 0$ in Eqs. (7), (9), and (10), which then gives

$$P_b = 0, \quad \sigma_b = 0, \quad \delta = \alpha_b(\Delta T)L.$$

8 Examples of statically indeterminate rod systems

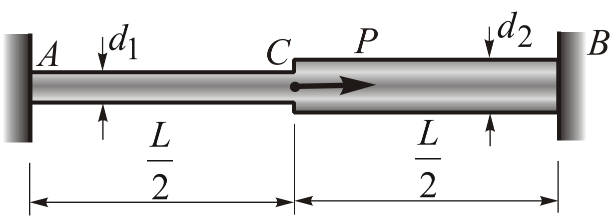


Fig. 29

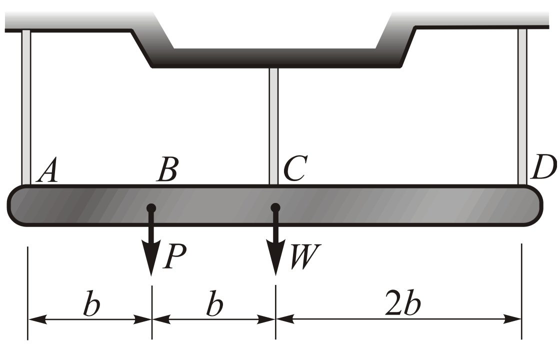


Fig. 30

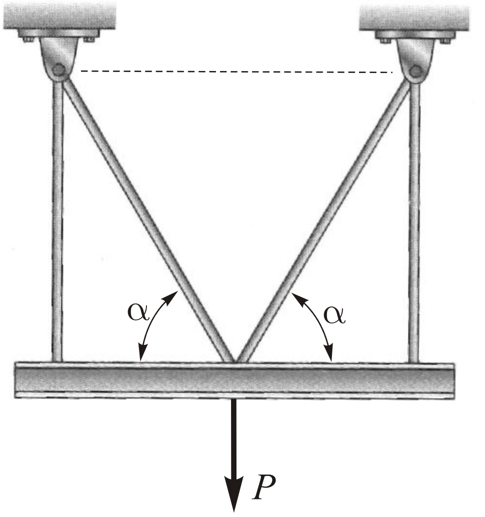


Fig. 31

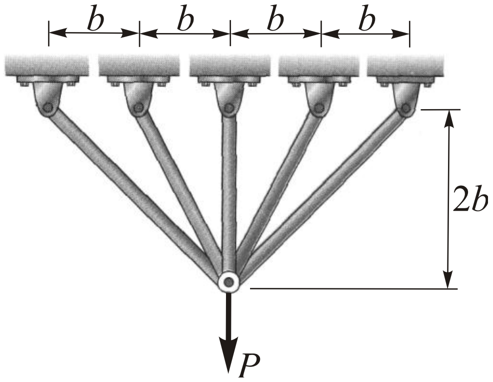


Fig. 32

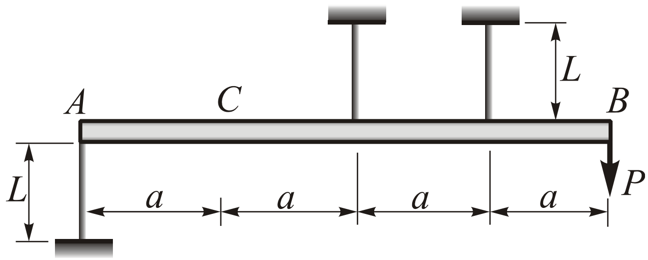


Fig. 33

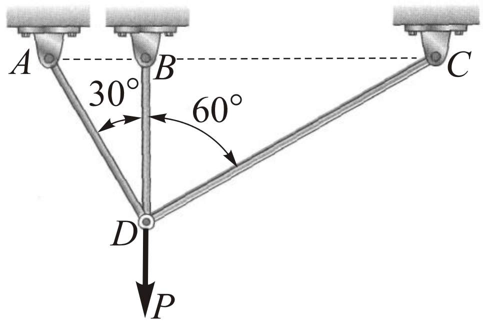


Fig. 34

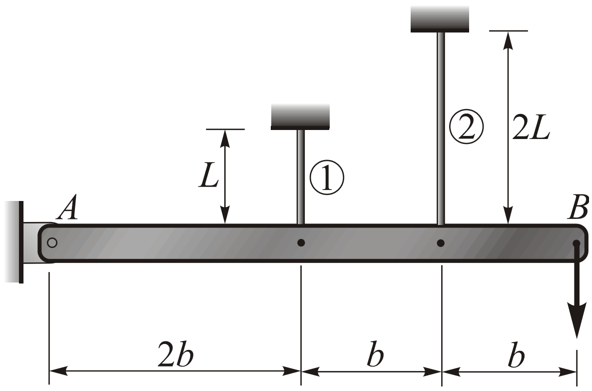


Fig. 35

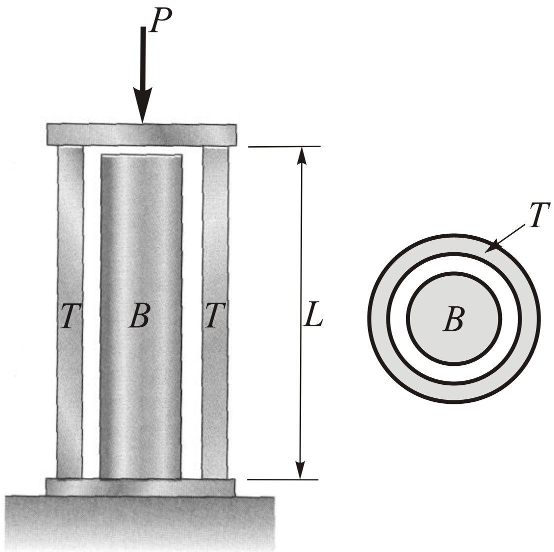


Fig. 36

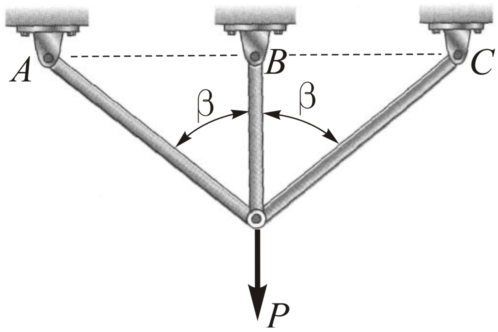


Fig. 37

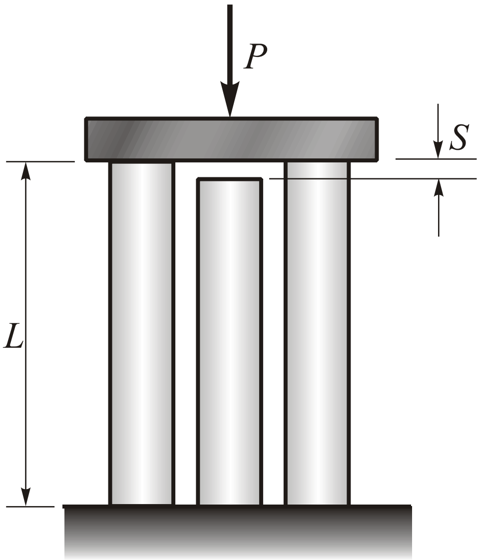


Fig. 38

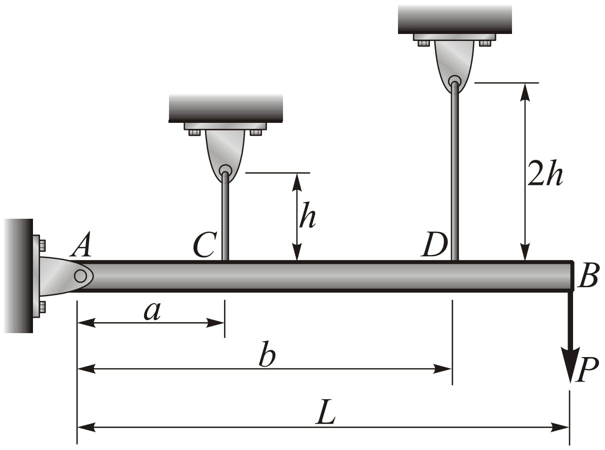


Fig. 39

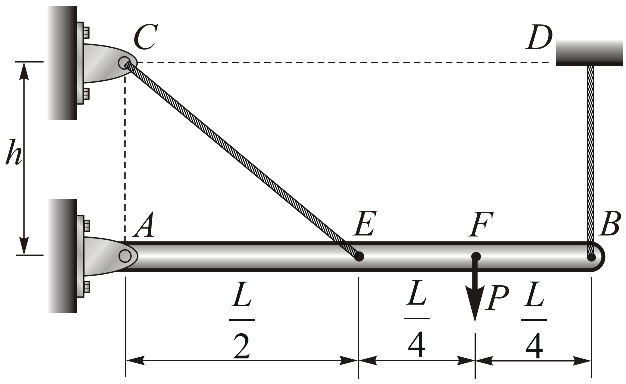


Fig. 40

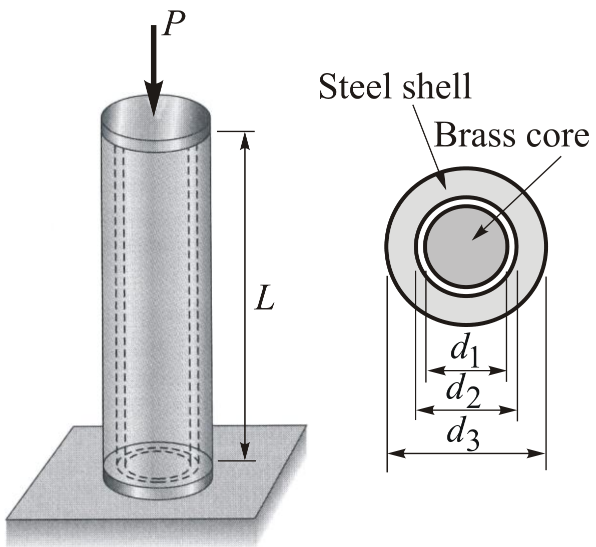


Fig. 41

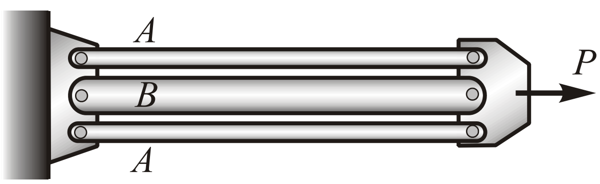


Fig. 42

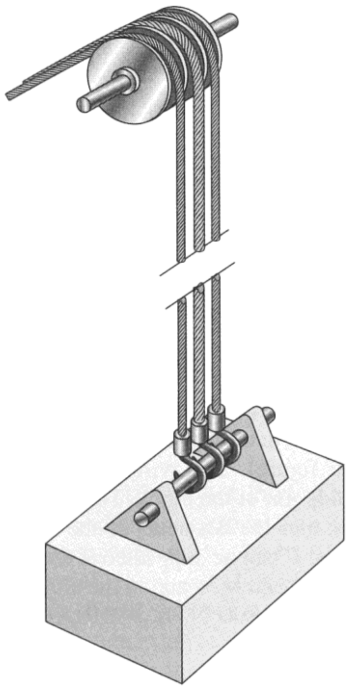


Fig. 43

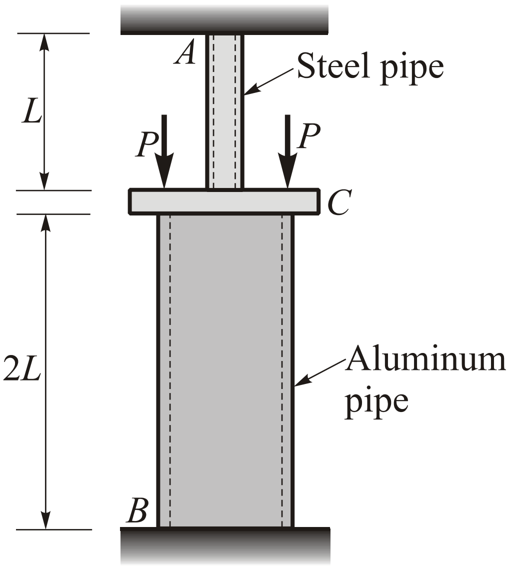


Fig. 44

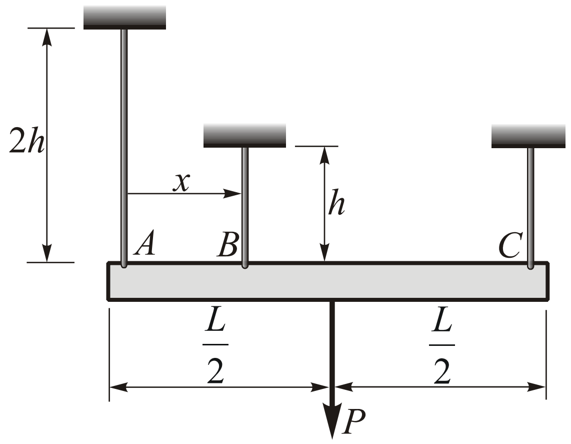


Fig. 45

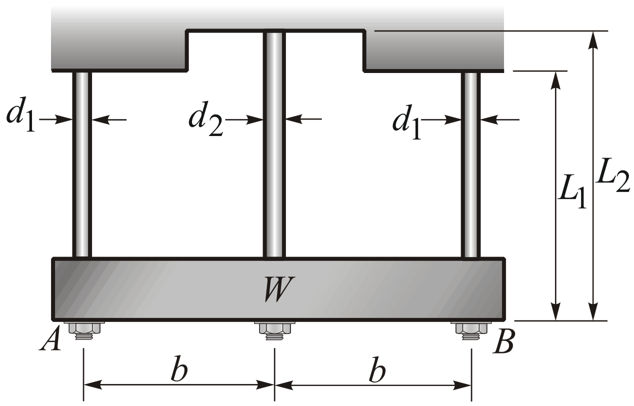


Fig. 46

MINISTRY OF EDUCATION AND SCIENCE OF UKRAINE

National aerospace university “Kharkiv Aviation Institute”

Department of aircraft strength

Course
Mechanics of materials and structures

HOME PROBLEM 7

Design problem for statically indeterminate rod in tension-compression

Name of student:

Group:

Advisor:

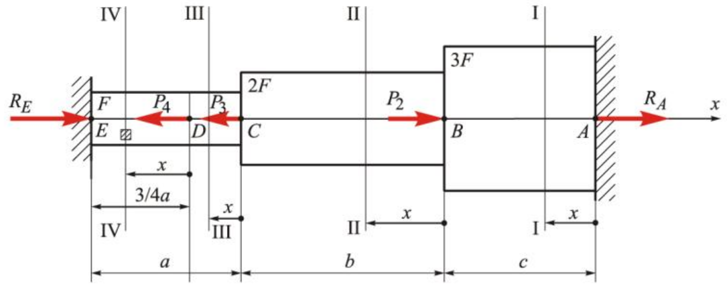
Data of submission:

Mark:

National aerospace university
“Kharkiv Aviation Institute”
Department of aircraft strength

Subject: mechanics of materials
Document: home problem
Topic: Stresses and elongations in statically indeterminate rods in tension-compression
Full name of the student, group

Variant: 1 **Complexity: 1**



Given: $[\sigma]_t = 160$ MPa; $[\sigma]_c = 200$ MPa;
$P_2 = 10$ kN; $P_3 = 50$kN, $P_4 = 80$kN;
$a = 3$ m, $b = 4$ m, $c = 5$ m.
Goal:
1) open static indeterminacy and design the graph on normal forces;
2) calculate cross-sectional area F;
3) calculate acting stresses in the portions of the rod and design the graph of their distribution along the length of the rod;
4) design the graph of the rod elongations;
5) estimate stress state in critical cross-section.

Full name of the lecturer **signature**

Mark: **Mark:**

General method in statically indeterminate rods and rod system analysis is in finding complementary equations of deformation compatibility to determine internal forces in the rod. The number of compatibility equations depends on degree of static indeterminacy.

In our case, degree of static indeterminacy $k = m - n$, where $m = 2$ – total number of constraints (reactions), $n = 1$ – number of equations of equilibrium.

After substituting

$k = 2 - 1 = 1$.

Conclusion: the rod is singly statically indeterminate.

Due to axial loading, only axial reactions R_A and R_E take place in this problem.

Solution

1. Calculating the support reactions R_A and R_E (see Fig. 1).

(a) from condition of equilibrium $\sum F_x = 0$. Direction to the right is assumed to be positive (see Fig. 1).

$$\sum F_x = 0 = R_E - P_4 - P_3 + P_2 + R_A = 0 .$$

(b) designing the compatibility equation.

It is evident that this deformable rod has two immobile cross-sections *A* and *E*. Therefore total elongation of the rod is zero, i.e. $\Delta l_{AE} \equiv 0$:

$$\Delta l_{AE} = \Delta l_{AB} + \Delta l_{BC} + \Delta l_{CD} + \Delta l_{DE} .$$

The elongations of particular portions *AB*, *BC*, *CD*, *DE* are generated by corresponding normal forces. According to the method of sections the equations of normal forces are the following:

I–I $(0 < x < c)$

$$N_x^I(x) = +R_A .$$

II–II $(0 < x < b)$

$$N_x^{II}(x) = +R_A + P_2 .$$

III–III $(0 < x < a/4)$

$$N_x^{III}(x) = +R_A + P_2 - P_3 .$$

IV–IV $(0 < x < 3a/4)$

$$N_x^{IV}(x) = +R_A + P_2 - P_3 - P_4 .$$

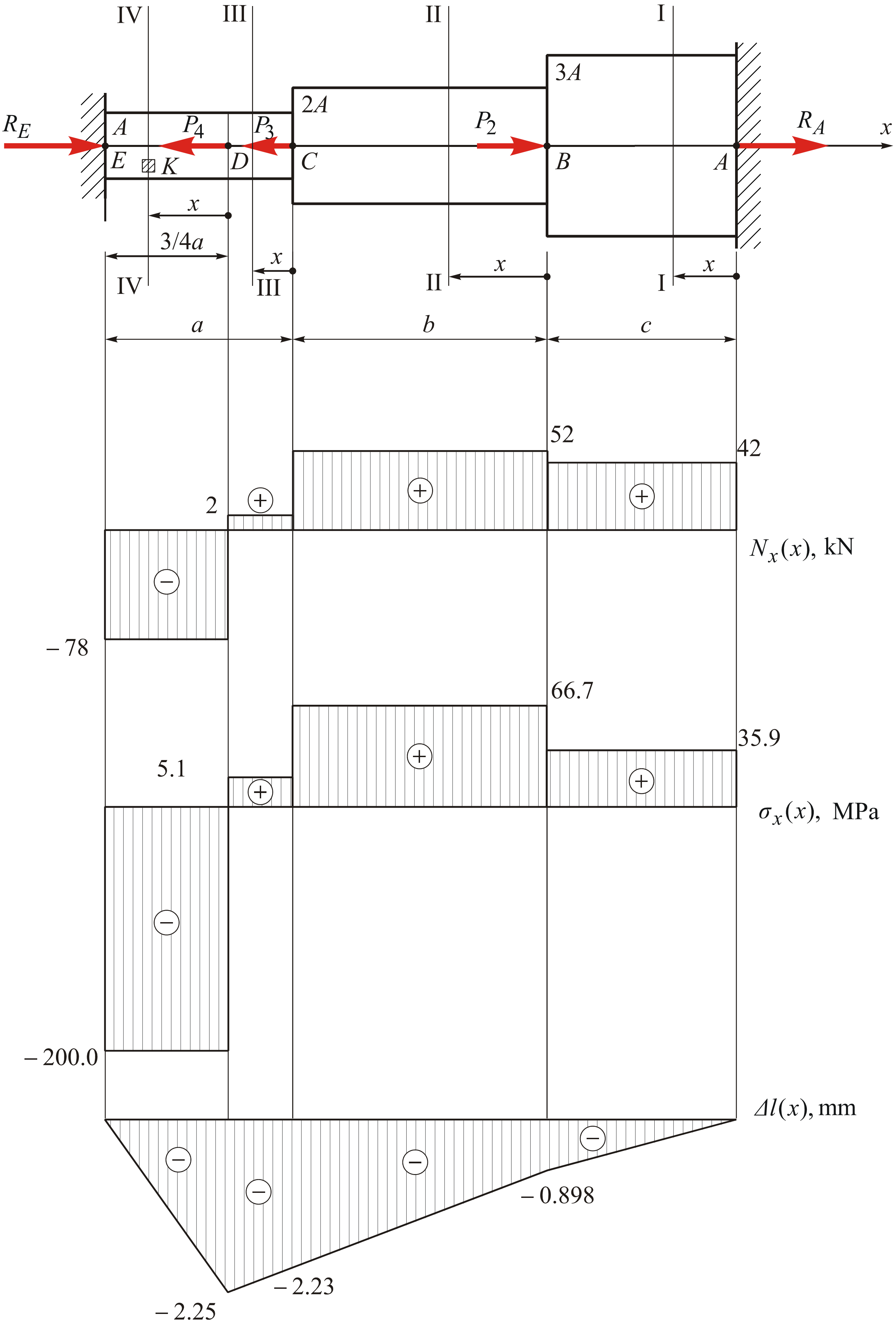

IV
III
II
I
3A
2A
R_E
A
P_4
P_3
P_2
R_A
x
E
K
D
C
B
A
x
3/4a
x
x
x
IV
III
II
I
a
b
c
52
42
2
– 78
$N_x(x)$, kN
66.7
35.9
5.1
– 200.0
$\sigma_x(x)$, MPa
$\Delta l(x)$, mm
– 0.898
– 2.23
– 2.25

Fig. 1

Corresponding elongations of the portions are:

$$\Delta l_{AB}=\frac{N_x^I(x)(c)}{3AE};\quad \Delta l_{BC}=\frac{N_x^{II}(x)(b)}{2AE};\quad \Delta l_{CD}=\frac{N_x^{III}(x)\left(\frac{1}{4}a\right)}{AE};$$

$$\Delta l_{DE}=\frac{N_x^{IV}(x)\left(\frac{3}{4}a\right)}{AE};$$

$$\to\frac{5}{3}R_A+2(R_A+P_2)+\frac{3}{4}(R_A+P_2-P_3)+\frac{9}{4}(R_A+P_2-P_3-P_4)=0\to$$

$$\to 80R_A=-60P_2+36P_3+27P_4=-600+1800+2160=3360\to R_A=\frac{3360}{80}=+42\ \text{kN.}$$

"+" sign of R_A reaction supports the conclusion that R_A reaction is actually determined to the right. After R_A finding, static indeterminacy is opened and normal forces may be determined.

2. Calculating the normal forces in an arbitrary cross-section of each portion.

$$N_x^I(x)=R_A=+42\ \text{kN},$$

$$N_x^{II}(x)=+R_A+P_2=+42+10=52\ \text{kN},$$

$$N_x^{III}(x)=R_A+P_2-P_3=+42+10-50=+2\ \text{kN},$$

$$N_x^{IV}(x)=R_A+P_2-P_3-P_4=+42+10-50-80=-78\ \text{kN}.$$

The graph of normal force distribution is shown on Fig. 1.

3. Calculating the cross-sectional area *A* from condition of strength in critical portion.
Due to $[\sigma]_t\neq[\sigma]_c$, it will be necessary to design two conditions of strength – for critically tensile and critically compressed portions. In our case,
(a) for three tensile portions *II-II* portion is evidently critical after comparing the relations $\frac{42}{3A}$, $\frac{52}{2A}$ and $\frac{2}{A}$. That is why

$$\sigma_{\max t}=\sigma_x^{II}=\frac{N_x^{II}(x)}{2A_t}\le[\sigma]_t\to A_t=\frac{N_x^{II}(x)}{2[\sigma]_t}=\frac{52\times10^3}{2\times160\times10^6}=1.625\times10^{-4}\ \text{m}^2;$$

(b) for compressed portion:

$$\left|\sigma_{\max c}\right|=\left|\sigma_x^{IV}\right|=\frac{\left|N_x^{IV}(x)\right|}{A_c}\le[\sigma]_c\to A_c=\frac{N_x^{IV}(x)}{[\sigma]_c}=\frac{78\times10^3}{200\times10^6}=3.9\times10^{-4}\ \text{m}^2.$$

For future calculating, we should select larger of two cross-sectional areas which will satisfy both conditions of strength:

$$A_{\max}=A_c=3.9\times10^{-4}\ \text{m}^2.$$

4. Calculating the acting stresses.

$$\sigma_x^I=\frac{N_x^I(x)}{3A_{\max}}=\frac{42\times10^3}{3\times3.9\times10^{-4}}=+35.9\ \text{MPa},$$

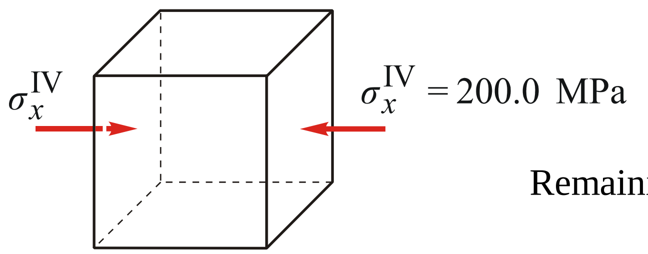
$$\sigma_x^{II}=\frac{N_x^{II}(x)}{2A_{\max}}=\frac{52\times10^3}{2\times3.9\times10^{-4}}=+66.7\ \text{MPa},$$

$$\sigma_x^{III} = \frac{N_x^{III}(x)}{A_{max}} = \frac{+2\times10^3}{3.9\times10^{-4}} = +5.1 \text{ MPa},$$

$$\sigma_x^{IV} = \frac{N_x^{IV}(x)}{A_{max}} = \frac{-78\times10^3}{3.9\times10^{-4}} = -200.0 \text{ MPa}.$$

Graph of stress distribution is shown on Fig. 1.

5. Analysis of stress state type in an arbitrary point K of critical section.



$\sigma_x^{IV} = \sigma_3 = -200$ MPa,

Remaining principal stresses are:

$\sigma_1 = \sigma_2 = 0.$

Fig. 2

Conclusion: stress state is uniaxial, deformation is tension.

6. Drawing the graph of displacements.

E point is selected as the origin. The displacements of particular points are designated by δ. Therefore, $\delta_E = 0$. Due to Hook's law, the displacement function is linear, that is why the displacements of each portion tip are numerically equal to the portion elongation or shortening. Elasticity modulus value $E = 2\times10^{11}$ Pa is used in this calculation.

$$\delta_D = \Delta l_{ED} = \frac{N_x^{IV}(x)3a}{4EA_{max}} = \frac{-78\times10^3\times3\times3}{4\times2\times10^{11}\times3.9\times10^{-4}} = -22.5\times10^{-4} \text{ m} = -2.25 \text{ mm}.$$

$$\delta_C = \Delta l_{EC} = \Delta l_{ED} + \Delta l_{DC} = -22.5\times10^{-4} + \frac{N_x^{III}(x)a}{4EA_{max}} = -22.5\times10^{-4} +$$

$$+\frac{2\times10^3\times3}{4\times2\times10^{11}\times3.9\times10^{-4}} = -22.5\times10^{-4} + 0.19\times10^{-4} = -22.31\times10^{-4} \text{ m} = -2.23 \text{ mm}.$$

$$\delta_B = \Delta l_{EB} = \Delta l_{EC} + \Delta l_{CB} = -22.31\times10^{-4} + \frac{N_x^{II}(x)b}{2EA_{max}} = -22.31\times10^{-4} +$$

$$+\frac{52\times10^3\times4}{2\times2\times10^{11}\times3.9\times10^{-4}} = -22.31\times10^{-4} + 13.33\times10^{-4} = -8.98\times10^{-4} \text{ m} = -0.898 \text{ mm}.$$

$$\delta_A = \Delta l_{EA} = \Delta l_{EB} + \Delta l_{BA} = -8.98\times10^{-4} + \frac{N_x^{I}(x)c}{3EA_{max}} = -8.98\times10^{-4} +$$

$$+\frac{42\times10^3\times5}{3\cdot2\times10^{11}\times3.9\times10^{-4}} = -8.98\times10^{-4} + 8.97\times10^{-4} = 0.01\times10^{-4} \text{ m}.$$

Let us estimate the error of calculating, since really the displacement of A point must be zero according to compatibility equation:

$$\Delta = \frac{0.01\times10^{-4}}{8.97\times10^{-4}}\times100\% = 0.11\%.$$

Due to negligibly little error, the calculation is true.

The graph of displacements is also shown on Fig. 1.